[Draft]

King Saud University
College of Sciences
Department of Mathematics
Semester 462 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Calculators are not allowed.***

Question 1 [Marks 10]**:** Which of the given choices is correct?

(i) If square of a matrix A is zero matrix, then $I - A$ is equal to:
a) 0 b) $(A-I)^{-1}$ c) $(A+I)^{-1}$ d) $A+I$

(ii) If A is a square matrix of order 3 with $det(A) = 2$, then $det(det(det(\frac{1}{det(A)}A^2)\,A^3)\,A^{-1})$ is equal to:
a) 1/4 b) ½ c) 1/3 d) 1/16

(iii) If the general solution of $AX = 0$ is $(-2r, 4r, r), r \in \mathbb{R}$, and $(1,0,-2)$ is a solution of $AX = B$, then the general solution of $AX = B$ is:
a) $(1-2r, 4r, r-2)$ b) $(-2r, 0, -2r)$ c) $(-2r, 4r, r)$ d) $(-2r-1, 4r, r-2)$

(iv) A subset $\boldsymbol{S}$ of $\mathbb{R}^3$ is a basis of the vector space $\mathbb{R}^3$ if $\boldsymbol{S}$ is equal to:
a) {(1,0,0),(0,2,1),(0,6,0)} b) {(1,1,0),(2,1,0),(3,2,0)} c) {(1,1,0),(0,0,0),(3,2,1)} d) {(1,1,0),(0,0,1),(2,2,1)}

(v) If $B = \{u_1 = (2,1), u_2 = (4,3)\}$ and $B' = \{u_1' = (0,1), u_2' = (6,0)\}$ are ordered bases of $\mathbb{R}^2$, then the transition matrix $\boldsymbol{P}_{B'\to B}$ from B' to B is equal to:
a) $\begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 9 \\ 1 & -3 \end{bmatrix}$ c) $\begin{bmatrix} -2/3 & 3 \\ 1/3 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

(vi) If B is a 3×3 matrix with $det(B) = 2$, then $nullity(B)$ is equal to:
a) 2 b) 1 c) 3 d) 0

(vii) If $\langle\,,\rangle$ is an inner product on $\mathbb{R}^n$ and $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^n$ such that $\|u\|^2 = 5$, $\|v\|^2 = 1$, $\langle u,v\rangle = -2$, then $\langle u + 2v, 5u - v\rangle$ is equal to:
a) $\sqrt{5}$ b) 5 c) 9 d) 41

(vii) If $S = \{A, I_2\} \subseteq M_{2\times2}(\mathbb{R})$, where A is a non-symmetric matrix, then S must be:
a) linearly dependent b) a spanning set for $M_{2\times2}(\mathbb{R})$ c) linearly independent d) orthogonal

(viii) Let $\boldsymbol{T}$ be the transformation from the Euclidean space $\mathbb{R}^2$ to $\mathbb{R}$ given by $\boldsymbol{T}(\boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{x} \in \mathbb{R}^2$, where $\|\boldsymbol{u}\|$ is the Euclidean norm of $\boldsymbol{u}$. Then, for $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ and $k \in \mathbb{R}$, $\boldsymbol{T}$ satisfies:
a) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ b) $T(\boldsymbol{u}+\boldsymbol{v}) \le T(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ c) $T(\boldsymbol{0}) > 0$ d) $T(k\boldsymbol{u}) = kT(\boldsymbol{u})$

(x) Zero is an eigen value of the matrix $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ with geometric multiplicity equal to:
a) 1 b) 2 c) 3 d) 4

Question 2 [Marks $2+2+3$]**:**

(a) Find the square matrix A of size 3 such that $A^{-1}(A-I)=\begin{bmatrix}1&1&1\\2&1&1\\1&1&2\end{bmatrix}$ and evaluate $det(A)$.

(b) Let $A=\begin{bmatrix}1&0&-1\\1&1&-2\\-2&-1&2\end{bmatrix}$ and $B=\begin{bmatrix}-2&1&1\\-1&1&-2\\1&-1&-2\end{bmatrix}$. Find a matrix X that satisfies $XA=B$.

(c) Solve the following system of linear equations by the Gauss elimination method:

$$\begin{aligned} x+\ y+\ z&=1\\ 2x\qquad +2z&=3\\ 3x+5y+4z&=2. \end{aligned}$$

Question 3 [Marks $3+3+3$]**:**

Let $A=\begin{bmatrix}1&0&1\\1&2&3\\2&0&2\end{bmatrix}$ and $B=\begin{bmatrix}1&0&1\\0&2&2\\0&0&0\end{bmatrix}$. Then:

(a) Find a basis and the dimension for each of the vector spaces *row*(A), *col*(A), and ***N***(A).

(b) Decide with justification whether the following statements are true or false:

(i) $row(A)=row(B)$ (ii) $col(A)=col(B)$ (iii) ***N***(A).= ***N***(***B***).

(c) Find an orthonormal basis relative to the Euclidean inner product for each of the spaces *row*(A), *col*(A), and ***N***(A).

Question 433 [Marks $3+(1+2)$]**:**

(a) Construct an orthogonal basis C of the Euclidean space $\mathbb{R}^3$ by apply the Gram-Schmidt algorithm on the given basis $B=\{v_1=(1,1,0), v_2=(1,0,1), v_3=(0,1,1)\}$, and then find the coordinate vector of $v=(1,1,0)\in\mathbb{R}^3$ relative to the orthogonal basis C.

(b) Let $\mathscr{P}_2$ denote the vector space of real polynomials with degree ≤ 2. Consider the linear transformation $\boldsymbol{T}:\mathbb{R}^3\to\mathscr{P}_2$ defined by: $\boldsymbol{T}(1,0,0)=x^2+1, \boldsymbol{T}(0,1,0)=3x^2+2, \boldsymbol{T}(0,0,1)=-x^2$. Then:

(i) Compute $\boldsymbol{T}(a,b,c)$, for all $(a,b,c)\in\mathbb{R}^3$.

(ii) Find a basis for each of the vector spaces $ker(\boldsymbol{T})$ and $Im(\boldsymbol{T})$.

Question 5 [Marks $2+3+3$]**:** Let $A=\begin{bmatrix}2&2&-2\\2&1&-1\\2&2&-2\end{bmatrix}$. Then:

(a) Find the eigenvalues of A.

(b) Find algebraic and geometric multiplicities of all the eigenvalues of A.

(c) Is the matrix A diagonalizable? If yes, find a matrix P that diagonalizes A.

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