Chapter 1 Introduction

1.1 Blasius Equation

If a fluid flows past a solid, a fluid layer is formed adjacent to the boundary of the solid. This layer is called a boundary layer and strong viscous effects exist within this layer.

Consider a uniform flow over a flat surface, \( y = 0, \ x \geq 0, \ -\infty < z < \infty \).

Equations of the flow in the boundary layer are the continuity equation

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

and the reduced Navier-Stokes equation

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}
\]

where \( u \) and \( v \) are respectively the components of the velocity vector and \( \nu \) represents the viscosity of the fluid.

Boundary conditions are

\[
u (x,0) = 0 \quad x \geq 0 \quad (1.3a)
\]
\[
u (x,0) = 0 \quad y \geq 0 \quad (1.3b)
\]
\[
u (x,y) \rightarrow U \quad \text{as} \quad y \rightarrow \infty \quad (1.3c)
\]

where \( U \) is the constant speed of the flow outside the boundary layer.

Define a stream function \( \psi(x,y) \) such that

\[
u u = \frac{\partial \psi}{\partial y}, \quad \nu = -\frac{\partial \psi}{\partial x}
\]

then equation (1.1) is satisfied identically and equation (1.2) becomes

\[
u \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \nu \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} + \nu \frac{\partial^3 \psi}{\partial y^3}
\]

Blasius used a similarity transformation to reduce (1.5) to an ordinary differential equation.
A similarity transformation is based on the symmetry analysis of a differential equation [11,23]. When a symmetry property of a differential equation is identified it can be exploited to achieve a simplification. If it is an ordinary differential equation then usually the order of the equation can be reduced. If it is a partial differential equation then usually the dependent and independent variables can be combined to achieve a reduction of order or a reduction of the partial differential equation to an ordinary differential equation.

In the case of (1.5) symmetry analysis leads to the following transformation [11]

\[ \eta = a \frac{y}{\sqrt{x}} \]

\[ \psi(x, y) = b \sqrt{x} f(\eta) \]

where a and b are constants and are chosen to make \( \eta \) and \( f(\eta) \) dimensionless. They are taken as

\[ a = \frac{U}{\sqrt{v}} \]

\[ b = \sqrt{vU} \]

With this choice, \( \eta \) is called the dimensionless similarity variable and \( f(\eta) \) is called the dimensionless stream function. Now

\[ \frac{\partial \psi}{\partial x} = -\frac{U}{2} \frac{y}{x} f'(\eta) + \frac{1}{2} \sqrt{vU} f(\eta) \frac{\sqrt{x}}{\sqrt{x}} \]

\[ \frac{\partial \psi}{\partial y} = U f'(\eta) \]

\[ \frac{\partial^2 \psi}{\partial y^2} = U f''(\eta) \frac{a}{\sqrt{x}} \]

\[ \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{U}{2} \frac{U}{\sqrt{v}} \frac{y^x}{x^2} f''(\eta) = -\frac{U}{2x} \eta f''(\eta) \]

\[ \frac{\partial^3 \psi}{\partial y^3} = \frac{U^2}{vx} f''''(\eta). \]
A substitution of the above derivatives in equation (1.5) reduces it to
\[
\frac{d^3 f}{d \eta^3} + \frac{1}{2} f (\eta) \frac{d^2 f}{d \eta^2} = 0
\]  
(1.6)

Equation (1.6) is known as the Blasius equation. The boundary condition (1.3a) transforms to
\[ f'(0) = 0 \]
(1.7a)
while (1.3b) becomes
\[ f(0) = 0 \]
(1.7b)
and (1.3c) reduces to
\[ f'(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty \]
(1.7c)
Equation (1.6) together with the boundary conditions (1.7a), (1.7b) and (1.7c) is called the Blasius problem.

1.2 Numerical solution

A numerical solution of the Blasius problem usually uses the shooting method. In this method it is assumed that
\[ f''(0) = \sigma \]
(1.8)
and the problem is solved with different values of \( \sigma \). Such values of \( \sigma \) will lead to different values of \( \frac{df}{d\eta} \) as \( \eta \rightarrow \infty \). We seek that value of \( \sigma \) which will yield an \( f \) which satisfies
\[ \lim_{\eta \rightarrow \infty} \frac{df}{d\eta} = 1. \]

First accurate numerical solution was obtained by Howarth [17]. More recently Asaithambi [8], and Cortell[12] have also solved the Blasius problem by the shooting method.
In practice it is impossible to carry out calculations up to infinity. Hence an $\eta = \eta_\infty$ is arbitrarily fixed and we demand that
\[
\frac{df}{d\eta} \approx 1 \text{ when } \eta = \eta_\infty.
\]

We solve the Blasius equation by the shooting method. We start with $\alpha = 0.1$ and find that $f'(\eta)$ between $\eta = 12$ and $13$ is practically a constant $= 0.449287$. With $\alpha = 0.6$. We find the slope for $12 < \eta < 13$ equal to 1.48352. This means the actual value should lie between 0.1 and 0.6. Therefore our next choice is $\frac{0.1 + 0.6}{2} = 0.35$ and so on. We present the results in Table 1 and 2.
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<thead>
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<th>$\alpha$</th>
<th>$f'(12.5)$</th>
</tr>
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<td>0.449287</td>
</tr>
<tr>
<td>0.6</td>
<td>1.48352</td>
</tr>
<tr>
<td>0.35</td>
<td>1.03571</td>
</tr>
<tr>
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<td>0.771459</td>
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<td>0.908413</td>
</tr>
<tr>
<td>0.31875</td>
<td>0.973101</td>
</tr>
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<td>1.00465</td>
</tr>
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<td>0.988937</td>
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Table 1: Sequence of values of $\alpha$ converging to $\sigma$
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<tr>
<th>$\eta$</th>
<th>$f(\eta)$</th>
<th>$f'(\eta)$</th>
<th>$f''(\eta)$</th>
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<tr>
<td>0.0</td>
<td>0</td>
<td>0</td>
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</tr>
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<td>0.0664081</td>
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<td>0.02656</td>
<td>0.132765</td>
<td>0.331468</td>
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<td>0.059735</td>
<td>0.198938</td>
<td>0.330081</td>
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<td>0.106109</td>
<td>0.264711</td>
<td>0.327391</td>
</tr>
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<td>0.165573</td>
<td>0.329782</td>
<td>0.323009</td>
</tr>
<tr>
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<td>0.23795</td>
<td>0.393778</td>
<td>0.31659</td>
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<td>0.456264</td>
<td>0.307867</td>
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<td>0.574761</td>
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<td>0.629769</td>
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<td>$f'\eta$</td>
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<td></td>
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<td>$3.28575 \times 10^{-6}$</td>
</tr>
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</table>

Table 2: Numerical values of $f(\eta), f'(\eta), f''(\eta)$
Fig1: $f(\eta)$ as a function of $\eta$

Fig2: $f'(\eta)$ as a function of $\eta$
Fig3: $f''(\eta)$ as a function of $\eta$
1.3 Analytical methods

Blasius [10] found a series solution of Eq.(1.6)

\[ f(\eta) = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \frac{A_k \sigma^{k+1}}{(3k + 2)!} \eta^{3k+2} \] (1.9)

where \( A_0 = A_1 = 1 \) and

\[ A_k = \sum_{r=0}^{k-1} \frac{3k-1}{3r} A_r A_{k-r-1}, \quad k \geq 2 \] (1.10)

However the series (1.9) converges slowly and its radius of convergence is approximately \( 5.90\sigma \), therefore it cannot be used to find an accurate value of \( \sigma \) by using the condition

\[ f'(\eta_\infty) = 1 \]

for sufficiently large \( \eta_\infty \). Blasius found an asymptotic expression and estimated \( \sigma \) by matching the two expressions for a suitable value of \( \eta \).

In recent years a few authors have tried to find alternate methods to find \( \sigma \). J. H. He has proposed a perturbation approach to solve this problem [16] and Abbasbandy has combined this approach with Adomian decomposition method to find an improved solution of the problem [3]. Liao [19] has used the homotopy analysis method to find an accurate value of \( \sigma \).

Recently Wang has used a transformation to transform the Blasius equation into a second order differential equation in which the condition at infinity is transformed into

\[ \lim_{x \to 1} y(x) = 0 \] (1.11)

[24]. Several authors have used Eq. (1.11) to find a good estimate of \( \sigma \),[9,15,24].
1.4 Iteration perturbation method of He

Following J. H. He [16] we construct an iteration formula for
Blasius equation

\[ f''_{n+1} + \frac{1}{2}f'_{n+1} f''_{n+1} = 0, \quad (1.12) \]

where we denote by \( f_n \) the nth approximate solution. We will use the
perturbation method to find approximately \( f_{n+1} \).

We begin with

\[ f_0 = 2b, \quad (1.13) \]

Substituting \( f_0 \) into (1.12) results in

\[ f_1'' + bf_1' = 0. \quad (1.14) \]

The solution of Eq. (1.14) is

\[ f_1(x) = \eta - \frac{1}{b} (1 - e^{-b\eta}), \quad (1.15) \]

where \( b \) is an unknown constant.

Substituting \( f_1 \) into (1.12), we obtain

\[ f_2'' + \frac{1}{2} \left[ \eta - \frac{1}{b} (1 - e^{-b\eta}) \right] f_2' = 0. \quad (1.16) \]

We re-write Eq. (1.16) in the form

\[ f_2'' + bf_2' + \frac{1}{2} \left[ \eta - \frac{1}{b} (1 - e^{-b\eta}) - 2b \right] f_2' = 0, \quad (1.17) \]

and embed an artificial parameter \( \epsilon \) in Eq.(1.17):

\[ f_2'' + bf_2' + \frac{1}{2} \epsilon \left[ \eta - \frac{1}{b} (1 - e^{-b\eta}) - 2b \right] f_2' = 0. \quad (1.18) \]

Supposing that the solution of Eq. (1.18) can be expressed as

\[ f_2 = f_2^{(0)} + \epsilon f_2^{(1)} + ..., \quad (1.19) \]

we have the following linear equations:
The solution of (1.20) is \( f_2^{(0)}(\eta) = \eta - (1 - e^{-b\eta})/b \), substituting it into (1.21) results in

\[
[f_2^{(1)}]'' + b[f_2^{(1)}]' = -\frac{1}{2}(b\eta + e^{-b\eta} - 2b^2 - 1)e^{-b\eta}
\]

(1.22)

The constant \( b \) can be identified by the following expression:

\[
\int_0^\infty e^{-b\eta}(b\eta + e^{-b\eta} - 2b^2 - 1)e^{-b\eta} d\eta = 0.
\]

(1.23)

So we have

\[
\frac{1}{4b} \Gamma(2) + \frac{1}{3b} - \frac{1 + 2b^2}{2b} = 0,
\]

(1.24)

which leads to the result

\[
b = \frac{1}{\sqrt{12}} = 0.28867.
\]

(1.25)

The exact solution of Eq. (1.22) is not required, the expression, Eq. (1.23), requires no terms of \( \eta^n e^{\beta\eta} \) \( n=1,2,3,\ldots \) in \( f_2^{(i)} \). So we can assume that the approximate solution of Eq. (1.22) can be expressed as

\[
f_2^{(1)}(\eta) = Ae^{-b\eta} + \frac{1}{8}e^{-2b\eta} + B,
\]

(1.26)

where \( A = -\frac{1}{4} \) and \( B = \frac{1}{8} \), which are identified from the initial conditions

\[
f_2^{(1)}(0) = [f_2^{(1)}]'(0) = 0.
\]

By setting \( \epsilon = 1 \), we obtain first-order approximate solution, i.e.,

\[
f_2(\eta) = f_2^{(0)}(\eta) + f_2^{(1)}(\eta)
\]

(1.27)
A highly accurate numerical solution of Blasius equation has been provided by Howarth [17], who gives the initial slope \( f''(0) = 0.332057 \) [17]. Comparing the approximate initial slope
\[
f''(0) = [f''(0)](0) + [f''(1)](0) = 0.3095,
\]
we find that the relative error is 6.8%.

We can improve the accuracy of approximate solution to an high-order by iteration technique. Substituting \( f_2 \) into (1.12), and by parallel operation, the constant \( b \) can be identified by the following relation:
\[
0.3062. (1.29)
\]
which leads to the result
\[
b = 0.3062. (1.29)
\]
It is obvious that \( f''(0) = b + 0.25b^2 = 0.3296 \) reaches a very high accuracy, the 0.73% accuracy is remarkably good.

1.5 Homotopy analysis method

Recently Liao has developed a method for solving nonlinear problems which does not depend on the existence of a small parameter, also it generalizes the ideas involved in the decomposition method of Adomian or the artificial small parameter method of Lyapunov [21]. This method has been used by Liao and other authors to find analytical expressions for several nonlinear problems.

Suppose the problem to be solved is
\[
N [u(r)] = f(r) \quad (1.30)
\]
Introduce a new variable \( q \) and consider the equation
\[
(1-q)L[\Phi(r,q) - u_0(r)] = hq \{N[\Phi(r,q)] - f(r)\} \quad (1.31)
\]
where \( L \) is a suitably chosen linear operator, \( u_0(r) \) is an initial approximate solution and \( h \) is a parameter to be chosen later. In (1.31) put \( q=0 \). We get
\[
\Phi(r,0) = u_0(r) \tag{1.32}
\]
If we let \( q=1 \) in (1.31), we get
\[
N[\Phi(r,1)] = f(r) \tag{1.33}
\]
Whose solution is the required function \( u(r) \). Thus as \( q \) varies continuously from 0 to 1, the initial approximate solution \( u_0(r) \) evolves to the desired solution \( u(r) \) of the problem Liao gives this reason for calling the method as the homotopy analysis method.

Define
\[
u_n(r) = \frac{1}{n!} \frac{\partial^n}{\partial q^n} \Phi(r,q) \bigg|_{q=0}.
\tag{1.34}
\]
then
\[
\Phi(r,q) = u_0(r) + \sum_{n=1}^{\infty} u_n(r)q^n. \tag{1.35}
\]
If the series converges at \( q=1 \), then
\[
u(r) = \Phi(r,1) = u_0(r) + \sum_{n=1}^{\infty} u_n.
\tag{1.36}
\]
The functions \( u_n(r) \) are found successively from the equations
\[
L[u_n(r)] = h\{N[u_0(r)] - f(r)\} \tag{1.37}
\]
when \( n=1 \) and
\[
L[u_n(r) - u_{n-1}(r)] = hR_n(r), \tag{1.38}
\]
when \( n \geq 2 \), where
\[
R_n(r) = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial q^{n-1}} N[\Phi(r,q)] \bigg|_{q=0} \tag{1.39}
\]
It has been shown by Liao [19] that if we choose \( h=-1 \) and \( L = L_0 \), the linear operator chosen for the Adomian method, then the solution
produced by the homotopy analysis method is identical to the one given by the Adomian method.

1.6 Liao's solution of the Blasius problem

Liao [18] wrote the Blasius equation in the form

\[(1-q)L[F(\eta,h,\beta,q) - f_0(\eta)] = qh\left(\frac{\partial^3 F(\eta,h,\beta,q)}{\partial \eta^3} + \frac{1}{2} F(\eta,h,\beta,q) \frac{\partial^2 F(\eta,h,\beta,q)}{\partial \eta^2}\right)\]  

(1.40)

where \( \eta \in [0, \infty) \), \( h \neq 0 \), \( \beta > 0 \), \( q \in [0,1] \)

with the boundary conditions

\[F(0,h,\beta,q) = F'(0,h,\beta,q) = 0, \quad F'(\infty,h,\beta,q) = 1\]  

(1.41)

Here \( h \) and \( \beta \) are free parameters to be chosen later. Note that at \( q=0 \), we have

\[F(\eta,h,\beta,0) = f_0(\eta)\]

and at \( q=1 \)

\[F(\eta,h,\beta,1) = f(\eta)\]

where the initial approximation \( f_0(\eta) \) can be chosen freely Liao chooses the linear operator \( L \) as

\[L = \frac{\partial^3}{\partial \eta^3} + \beta \frac{\partial^2}{\partial \eta^2}, \quad \beta > 0\]

and

\[f_0(\eta) = \eta - \frac{1-\exp(-\beta \eta)}{\beta}\]

as the initial approximation which satisfies the boundary conditions \( f(0)=f'(0)=0 \) and \( f'(\infty)=1 \). Successive elements of the series \( f_1(\eta,h,\beta), f_2(\eta,h,\beta), \ldots \ldots \) are calculated recursively and the solution is formally written as
Liao also finds a sequence of approximations for $f''(0, h, \beta)$ as

$$
\sigma_1 = \beta(1 + h) \frac{h}{4\beta}
$$

$$
\sigma_2 = \beta(1 + h)^2 \frac{1}{2 \beta} - \frac{5 h^2}{24 \beta^3}
$$

$$
\sigma_3 = \beta(1 + h)^3 \frac{3 h}{4 \beta} - \frac{5 h^2}{8 \beta^3} - \frac{275 h^3}{576 \beta^5} + \frac{5 h^3}{24 \beta^3}
$$

\vdots

He finds that with $\beta = 3$ and $h = -\frac{9}{10}, \sigma_{30} = 0.33205$ and $\sigma_{35} = 0.33206$. Thus $\beta = 3, h = -\frac{9}{10}$ are the appropriate values of these parameters since they yield the correct value of $f''(0)$ found by Howarth [17]. When we set $\beta = 3, h = -\frac{9}{10}$ and $q=1$ in (1.42) we find that the solution with the series truncated after 35 terms gives numerical values on $0 \leq \eta \leq 100$ which agree very well with the solution of Howarth [17].

1.7 Generalized Blasius problem

Cortell numerically studied the problem

$$
a f'''' + f f'' = 0 \quad (1.43a)
$$

$$
f(0) = f'(0) = 0, f'''(\infty) = 1 \quad (1.43b)$$

for various values of $a$ and observed that an increase in $a$ decreases $f(\eta)$ and $f'(\eta)$ also $f''(\eta)$ near $\eta = 0$ decreases at first with an increase in $a$ but further away it increases with $a$. Fang et al. [13] explained the above behavior analytically by making the following transformation

$$
\zeta = \frac{\eta}{\sqrt{a}}
$$

$$
f(\eta) = \sqrt{a} F(\zeta)
$$
Then the problem (1.43) transforms to
\[
\frac{d^3 F}{d \zeta^3} + F \frac{d^2 F}{d \zeta^2} = 0
\]  
(1.44)

\[
F(0) = \left. \frac{dF}{d \zeta} \right|_{\zeta=0} = 0, \left. \frac{dF}{d \zeta} \right|_{\zeta \to \infty} = 1
\]

whose solution can be easily obtained. They showed that
\[
\frac{\partial f}{\partial a} = \frac{1}{2\sqrt{a}} \left[ F(\zeta) - \zeta \frac{dF}{d \zeta} \right] 
\]  
(1.45)

\[
\frac{\partial f'}{\partial a} = -\frac{\eta}{2a^{3/2}} \frac{d^2 F}{d \zeta^2}
\]  
(1.46)

\[
\frac{\partial f''}{\partial a} = \frac{1}{2a^{3/2}} \frac{d^2 F}{d \zeta^2} [\zeta F(\zeta) - 1]
\]  
(1.47)

Now from (1.45) and (1.46) we see that \( \frac{\partial f}{\partial a} \leq 0, \frac{\partial f'}{\partial a} < 0 \), therefore \( f \) and \( f' \) decrease with an increase in \( a \). From (1.47) we see that the right side is negative to start with but becomes positive later on. This explains the behavior of \( f'' \) noted by Cortell [12].
Chapter 2 Wang's Equation

2.1 Introduction

The two dimensional constant laminar viscous flow over a semi-infinite flat plate is modeled by the nonlinear differential equation

\[ f''(\eta) + \beta f' f(\eta) f(\eta) = 0, \eta \in [0, \infty). \]  

(2.1a)

With the boundary conditions

\[ f(0) = f'(0) = 0, f'(\infty) = 1 \]  

(2.1b)

The above boundary value problem is called the Blasius problem [10] and it successfully describes the velocity distribution and other physical parameters associated with the fluid motion in the boundary layer along the plane [22]. The main hurdle in the solution of the above problem is the absence of the second derivative \( f''(0) \).

Once this derivative has been correctly evaluated an analytical solution of the boundary value problem may be readily found. Blasius [10] found the following power series solution of the problem with \( \beta_0 = \frac{1}{2} \)

\[ f(\eta) = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \]  

(2.2)

where \( A_0 = A_1 = 1 \) and

\[ A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}, \quad k \geq 2 \]

Where \( \sigma \) represents the unknown \( f''(0) \). Howarth [17] solved (1.1), (with \( \beta_0 = \frac{1}{2} \)), numerically and found

\[ \sigma = 0.33206 \]  

(2.3)

Several authors have devised numerical algorithms to find good approximations to \( f''(0) \) see, for example, Asaithambi [8] and references
therein. Asithambi [8] solved (2.1), (with $\beta_0=1$), and found $f''(0)$, denoted by $\alpha$, to be
\[ \alpha = 0.46900 \] (2.4)
Fang et al. [13] have shown that the substitution
\[ f(\eta) = \frac{1}{\sqrt{\beta_0}} F(\sqrt{\beta_0}\eta) \]
transforms Eq. (2.1a) into
\[ F'' + FF' = 0 \]
Therefore it is sufficient to consider the Blasius problem with $\beta_0=1$.
Henceforth we shall treat the problem with $\beta_0=1$. Liao [18] applied his homotopy analysis method to the Blasius problem and obtained the solution to a high level of accuracy. J. H. He [16] has used an iterative perturbation technique to find an approximate analytical solution of the Blasius problem. Abbasbandy [3] has used a modified version of the Adomian decomposition method to find a numerical solution while Cortell [12] has studied the dependence of the solution on the parameter $\beta_0$.
Recently Wang [24] has introduced a fresh approach to the Blasius problem. He used an ingenious idea to transform the Blasius problem into a simpler problem governed by a second order differential equation. He used the transformation
\[ x = f'(\eta), \quad y = f''(\eta) \] (2.5)
to transform (2.1) to
\[ \frac{d^2y}{dx^2} + \frac{x}{y} = 0, \quad x \in [0,1) \] (2.6a)
with the boundary conditions
\[ y(0) = f''(0), \quad y'(0) = 0, \quad \lim_{x \to 1} y(x) = 0 \] (2.6b)
Wang [24] used the Adomian decomposition method to solve (2.6), and found
\[
y(x) = \alpha - \frac{x^3}{6\alpha} - \frac{x^6}{180\alpha^2} - \frac{x^9}{2160\alpha^3} - \frac{x^{12}}{19008\alpha^4} \ldots. \tag{2.7}
\]

To find \(\alpha\) the equation \(y(1)=0\) is solved for \(\alpha\). He solved this equation retaining six terms of the series (2.7) and found \(\alpha = 0.453539\). Recently Hashim [15] improved this value to \(\alpha = 0.466799\) by finding terms of the series (2.7) up to \(x^{24}\) by the Adomian decomposition method (ADM) and then approximating this function by the [12/12] diagonal Pade’ approximating.

2.2 The Adomian decomposition method

Adomian's decomposition method is used for solving operator equations of the form
\[
u = \nu + Nu \tag{2.8}
\]
where \(N\) is a nonlinear differential operator and \(\nu\) is known. Since the method does not resort to linearization or assumptions of week nonlinearity, the nonlinearities which can be handled are quite general and the solutions generated may be more realistic than those achieved by simplifying the model of the physical problem to achieve conditions required for other techniques [14].

Adomian's method assumes that the solution \(u\) can be expanded as an infinite series:
\[
u = \sum_{n=0}^{\infty} u_n \tag{2.9}
\]
With \(u_n \in X, \forall n\). The image \(Nu\) has an expansion
\[
uu = \sum_{n=0}^{\infty} A_n \tag{2.10}
\]
With \( A_n \in X, \forall n \). Substituting (2.9) and (2.10) into (2.8) gives
\[
\sum_{n=0}^{\infty} u_n = v + \sum_{n=0}^{\infty} A_n
\] (2.11)
which is satisfied formally if we set
\[
u_0 = v \quad (2.12)
\]
\[
u_{n+1} = A_n \quad (2.13)
\]
Thus, we need to determine the so-called Adomian polynomials
\( A_n, n = 1, 2, 3,... \) To determine the \( A_n \)'s, a scalar parameter \( \lambda \) is introduced to set
\[
u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \] (2.14)
And
\[
u(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n \] (2.15)
As in [4,5], we can deduce that the coefficients \( A_n, n = 1, 2, 3,... \) are given by
\[
A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \nu(\lambda) \bigg|_{\lambda=0} \quad (2.16)
\]
Thus,
\[
A_0 = \nu(0),
\]
\[
A_1 = \frac{d^1}{d\lambda^1} \nu(\lambda) \bigg|_{\lambda=0} = (DN)(u_0)u_1,
\]
\[
A_2 = (DN)(u_0)u_2 + \frac{1}{2!}(D^2N)(u_0)u_1^2,
\]
\[\vdots\]
And so on, where \( D^{(r)}N(u_0) \) denotes the r-th derivative of \( N \) at \( u_0 \in X \). We should note that while \( A_n \) is a polynomial in \( u_0, u_1, u_2,... \), its dependence on \( u_0 \) is decided by the form of \( N \) and its derivatives. However generally, \( A_n \) depends only on \( u_0, u_1,..., u_n \) so that (2.16) allows successive calculation of
at any stage the solution can be approximated by the partial sum
\[ s_N = \sum_{j=0}^{N-1} u_j \]

\subsection{2.3 Wang's transformation}

Define \( x = f'(\eta) \), \( y = f''(\eta) \), then
\[
\frac{dy}{dx} = \frac{d}{d\eta}(f''(\eta)) \frac{d\eta}{dx} = f''''(\eta) \frac{d\eta}{dx} = f'''(\eta),
\]

Using this in (2.1a) we get
\[
f'''(\eta) \frac{dy}{dx} + \beta_0 f''''f''(\eta) = 0 \tag{2.17}
\]

Also assuming \( f''''(\eta) \neq 0 \), we can write (2.1a) as
\[
\frac{f''''(\eta)}{f'''(\eta)} = -\beta_0 f''(\eta) \tag{2.18}
\]

Integrate from 0 to \( \eta \).
\[
\left[ \ln(f''''(\eta)) \right]_0^\eta = -\beta_0 \int_0^\eta f'(s) ds
\]
\[
\ln\left( \frac{f''''(\eta)}{f''''(0)} \right) = -\beta_0 \int_0^\eta f'(s) ds
\]
\[
f''''(\eta) = f''''(0) \exp\left[ -\beta_0 \int_0^\eta f'(s) ds \right] \tag{2.19}
\]

Therefore \( f''''(\eta) > 0 \) if \( f'''(0) > 0 \). Our assumption that \( f''''(\eta) \neq 0 \) is justified.

We can cancel \( f''''(\eta) \) in Eq.(2.17). We get
\[
\frac{dy}{dx} + \beta_0 f''(\eta) = 0
\]

Differentiate again with respect to \( x \).
\[
\frac{d^2y}{dx^2} + \beta_0 f'(\eta) \frac{d\eta}{dx} = 0 \\
\text{or} \quad \frac{d^2y}{dx^2} + \beta_0 \frac{x}{y} = 0 \\
\text{or} \quad \frac{d^2y}{dx^2} + \beta_0 \frac{x}{y} = 0
\]

The above equation was obtained by Wang [24]. The new boundary condition are as follows

\[ x=0 \text{ corresponds to } \frac{df}{d\eta} = 0 \text{ which occurs at } \eta = 0. \text{ At this point} \]

\[ y = \frac{d^2f}{d\eta^2} = \alpha. \text{ Thus } y(0) = \alpha. \]

Also \( \frac{dy}{dx} = -\beta_0 f(\eta). \text{ At } \eta = 0, \text{ the right side will be zero giving us } y'(0) = 0. \]

The condition \( \lim_{\eta \to \infty} \frac{df}{d\eta} = 1 \) indicates that the domain over which Wang's equation holds is \([0,1). \text{ From (2.19) we conclude that}, \]

\[ \lim_{\eta \to \infty} f'^{(\eta)}(\eta) = 0 \]

This condition transforms to

\[ \lim_{x \to 1} y(x) = 0 \]

### 2.4 Solution of Wang's equation by Adomian decomposition method

\[ y' + \frac{x}{y} = 0, \quad 0 < x < 1, \quad \text{(2.20)} \]

Subject to the conditions

\[ y'(0) = 0, \quad y(1) = 0, \quad \text{(2.21)} \]

Along with the condition

\[ y(0) = \alpha \quad \text{(2.22)} \]
Following the standard procedure of the ADM [4], we write Eq. (2.20) in the operator form

\[ L y = N_y, \quad (2.23) \]

Where the linear operator \( L = \frac{d^2}{dx^2} \) is easily invertible \( N_y = -\frac{x}{y} \) represents the nonlinear term. Applying the inverse operator \( \int_0^x \int_0^x \) to both sides of Eq.(2.23) and incorporating the first condition of (2.21) and condition (2.22) yield

\[ y(x) = \alpha + L^{-1}(N_y) \quad (2.24) \]

The ADM defines the solution \( y(x) \) by the series: \( y = \sum_{n=0}^{\infty} y_n \)

The nonlinear term \( N_y \) is defined by:

\[ N_y = \sum_{n=0}^{\infty} A_n \]

where the Adomian polynomials \( A_n \) can be generated for all types of nonlinearities according to algorithm set by Adomian [4],

\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d \lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \ldots \quad (2.25) \]

In the ADM, the components \( y_n \) are determined by the recursive algorithm:

\[ y_0 = \alpha, \quad y_{k+1} = L^{-1}(A_k), \quad 0 \leq k \]

Some of the Adomian polynomials for \( N_y = -\frac{x}{y} \) calculated form (10) are
\[A_0 = -x \left( \frac{1}{y_0} \right),\]
\[A_1 = -x \left( \frac{-y_1}{y_0^2} \right),\]
\[A_2 = -x \left( \frac{y_1^2}{y_0^3} \frac{y_2}{y_0^2} \right),\]
\[A_3 = -x \left( \frac{-y_3}{y_0^5} + \frac{2y_1y_2}{y_0^3} - \frac{y_1^3}{y_0^4} \right),\]
\[\vdots\]

For numerical comparison purposes, we construct the solution
\[
y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots\]
as
\[
\lim_{n \to \infty} \phi_n(x) = y(x), \quad (2.26)
\]
Where \( \phi_n(x) \) is the n-term approximate solution. The theoretical treatment of the convergence of the ADM has been considered and discussed in [1,2]. Below we present first few members of the set of Adomian polynomials associated with this problem as well as the functions \( y_0, y_1, y_2, \ldots \) calculated from them.

\[
y_0 = \alpha, \quad A_0 = -\frac{x}{\alpha},
\]
\[
y_1 = -\frac{x^3}{6\alpha}, \quad A_1 = -\frac{x^4}{6\alpha^2},
\]
\[
y_2 = -\frac{x^6}{180\alpha^3}, \quad A_2 = -\frac{x^7}{30\alpha^3},
\]
\[
y_3 = -\frac{x^9}{2160\alpha^4}, \quad A_3 = -\frac{x^{10}}{144\alpha^4},
\]
\[
y_4 = -\frac{x^{12}}{19008\alpha^5}, \quad A_4 = -\frac{2099x^{13}}{1425600\alpha^5},
\]
\[
y_5 = -\frac{2099x^{15}}{2993766000\alpha^6}, \quad A_5 = -\frac{31453x^{16}}{99792000\alpha^7},
\]
\[\vdots\]

The value of \( \alpha \) can be found by using the last condition in (2.21), i.e. \( y(1)=0 \).
By solving $\phi_k(l) = 0, \quad k = 1, 2, \ldots, 6$, we find

\[
\begin{align*}
\alpha_1 &= 0.408248 \\
\alpha_2 &= 0.441743 \\
\alpha_3 &= 0.452576 \\
\alpha_4 &= 0.457674 \\
\alpha_5 &= 0.460566 \\
\alpha_6 &= 0.462404
\end{align*}
\]

2.5 Pade approximation

Hashim [15] improved the above values by approximating the function $y$ into a rational function with respect to the variable $x$ using diagonal Pade approximation. He employed the Maple built-in diagonal Pade approximants together with the condition $\phi_n(l) = 0$ to get the approximate value for $\alpha$.

He presented the calculated values for $\alpha$ using the $\phi_n$ for both the ADM and ADM-Pade approximation. The ADM with Pade approximation gives more accurate results compared with the standard ADM without Pade approximation. The accuracy of the solution will be further improved if more terms in the series solution are used. Hashim's results are reproduced in Table 3.

<table>
<thead>
<tr>
<th>Level of Pade approximation</th>
<th>Value of $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[3/3]</td>
<td>0.447214</td>
</tr>
<tr>
<td>[6/6]</td>
<td>0.463257</td>
</tr>
<tr>
<td>[9/9]</td>
<td>0.466791</td>
</tr>
<tr>
<td>[12/12]</td>
<td>0.466799</td>
</tr>
</tbody>
</table>

Table 3: Evaluation of $\alpha$ using Pade approximation
Chapter 3 Analytical Solution of Wang's Equation

3.1 Introduction

In this chapter we shall find an analytical solution of Wang's equation by a method independent of Adomian decomposition method. This method can easily extend the series (2.7) to as many terms as we wish. From the results in [15,24] it seemed that one could determine $\alpha$ to arbitrary accuracy once a large number of terms of the series (2.7) was available. Unfortunately this expectation turns out to be false. We shall derive two sequences one of them increasing the other decreasing, both of them converging to $\alpha$. The convergence is hopelessly slow. However the two sequences enable us to check accuracy of the approximation at each step.

We shall establish that the integral $\int_0^\infty \left( \frac{d^2f}{d\eta^2} \right)^2 d\eta$ is bounded below by $\frac{3}{4} \alpha$ and above by $\alpha$. If we use the value of $\alpha=0.469606$ then we get the exact result $\int_0^\infty \left( \frac{d^2f}{d\eta^2} \right)^2 d\eta = 0.37118$

We use the above result to find an approximate asymptotic expression for $f(\eta)$. The power series solution (2.2) is known to be valid for $0 \leq \eta \leq 5.69$ [18]. This solution combined with an asymptotic solution will represent the function $f(\eta)$ analytically on the entire domain $0 \leq \eta \leq \infty$.

3.2 Blasius solution

First we obtain Blasius solution give by (2.2) of the last chapter.

We want to solve

\begin{align*}
  f'''(\eta) + \beta_0 f''(\eta)f(\eta) &= 0 \\
  f(0) &= 0, \quad f'(0) = 0, \quad f''(0) = \sigma
\end{align*} \quad (3.1)
Let \( f(\eta) = \sum_{n=0}^{\infty} a_n \eta^n \).

Successive derivatives give us

\[
f'(\eta) = \sum_{n=1}^{\infty} n a_n \eta^{n-1},
\]

\[
f''(\eta) = \sum_{n=2}^{\infty} n(n-1) a_n \eta^{n-2},
\]

\[
f'''(\eta) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n \eta^{n-3},
\]

Put them in (3.1)

\[
\sum_{n=1}^{\infty} n(n-1)(n-2) a_n \eta^{n-3} + \beta_0 \sum_{k=2}^{\infty} k(k-1) a_k \eta^{k-2} \sum_{n=0}^{\infty} a_n \eta^n = 0
\]

The above expression is equivalent to

\[
\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} \eta^n + \beta_0 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (k+2)(k+1) a_{k+2} a_k \eta^{k+n} = 0
\]

Or

\[
\sum_{m=0}^{\infty} (m+3)(m+2)(m+1) a_{m+3} \eta^m + \beta_0 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (k+2)(k+1) a_{k+2} a_k \eta^{k+n} = 0
\] (3.2)

Define \( m = k + n \). Therefore \( n = m - k \).

Since \( n \geq 0 \), this implies \( 0 \leq k \leq m \). Eq. (3.2) becomes

\[
\sum_{m=0}^{\infty} (m+3)(m+2)(m+1) a_{m+3} \eta^m + \beta_0 \sum_{m=0}^{\infty} \sum_{k=0}^{m} (k+2)(k+1) a_{k+2} a_{m-k} \eta^m = 0
\]

Since each coefficient in the above series must vanish, we get

\[
(m+3)(m+2)(m+1) a_{m+3} + \beta_0 \sum_{k=0}^{m} (k+2)(k+1) a_{k+2} a_{m-k} = 0
\]

or

\[
a_{m+3} = -\frac{\beta_0}{(m+3)(m+2)(m+1)} \sum_{k=0}^{m} (k+2)(k+1) a_{k+2} a_{m-k}
\] (3.3)

Since \( a_0 = 0 \), \( a_i = 0 \), \( a_2 = \frac{\sigma}{2!} \), we get from (3.3)
\[
\begin{align*}
a_5 &= 0, \quad a_4 = 0, \quad a_3 = -\frac{\beta_0 \sigma^2}{5!} \\
a_6 &= 0, \quad a_7 = 0, \quad a_8 = \frac{11 \beta_0^2 \sigma^3}{8!} \\
a_9 &= 0, \quad a_{10} = 0, \quad a_{11} = -\frac{375 \beta_0^3 \sigma^4}{11!}
\end{align*}
\]

It is clear that we shall get non zero \(a_m\) for \(m=2,5,8,11,\ldots\)

Let \(m = 3\ell - 1, \quad \ell = 1, 2, 3,\ldots\)

\[
a_{3\ell+2} = -\frac{\beta_0}{(3\ell)(3\ell+1)(3\ell+2)} \sum_{k=0}^{3\ell-1} (k+2)(k+1)a_{k+2}a_{3\ell-k-1}, \quad k = 0, 3, 6, \ldots, 3\ell-3
\]

We shall get non zero coefficients if

\(k + 2 = 2, 5, 8,\ldots\), for this reason we define \(k + 2 = 3r + 2, \quad r = 0, 1, 2,\ldots\)

or \(k = 3r, \quad 0 \leq k \leq \ell - 1\)

Now Eq. (3.3) transforms to

\[
a_{3\ell+2} = -\frac{\beta_0}{(3\ell)(3\ell+1)(3\ell+2)} \sum_{r=0}^{3\ell-1} (3r+2)(3r+1)a_{3r+2}a_{3\ell-3r-1}, \quad (3.4)
\]

In order to get the solution in the form (2.2) we define

\[
a_{3\ell+2} = \frac{b_{3\ell+2} \sigma^{\ell+1}}{(3\ell+2)!}
\]

and replace \(a_{3\ell+2}, a_{3r+2}\) etc. in (3.4). We get

\[
b_{3\ell+2} \sigma^{\ell+1} = -\frac{\beta_0}{3\ell(3\ell+1)(3\ell+2)} \sum_{r=0}^{3\ell-1} (3r+2)(3r+1)b_{3r+2} \sigma^{r+1} b_{3\ell-3r-1} \sigma^{\ell-r}
\]

Simplification gives

\[
b_{3\ell+2} = -(3\ell-1)! \beta_0 \sum_{r=0}^{\ell-1} \frac{b_{3r+2} b_{3\ell-3r-1}}{3\ell(3\ell+1)(3\ell-3r-1)!}
\]

\[
= -\beta_0 \sum_{r=0}^{\ell-1} \frac{(3\ell-1)!}{3\ell} b_{3r+2} b_{3\ell-3r-1}
\]

(3.6)

From (3.5) we find

\[
a_2 = \frac{b_2}{2!} \sigma
\]
Since \( a_2 = \frac{\sigma}{2} \), we get \( b_2 = 1 \)

Formula (3.6) with \( b_2 = 1 \) produces the solution given by Blasius [10] in 1908.

### 3.3 Solution of Wang's equation

Wang's equation can be made a useful tool to elucidate analytical features of the Blasius problem. He solved this equation by Adomian decomposition method. However this method requires a lot of computation including double integration. In this section we shall solve Wang's equation by the same approach as was used to find Blasius solution. Re-write Eq. (2.6a) in the form

\[
y y'' = -x
\]  

(3.7)

We assume the solution in the form

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n
\]  

(3.8)

From the boundary conditions (2.6b), it is clear that

\[
a_0 = f'''(0), \quad a_1 = 0
\]  

(3.9)

We define \( \alpha = f'''(0) \). We are following the same notation as used by Wang [24] and Hashim [15], since in (2.1a) has now been put equal to unity.

Differentiate y twice and substitute \( y \) and \( y'' \) in (3.7). We get

\[
\sum_{k=0}^{\infty} \sum_{n=2}^{\infty} a_k a_n n(n-1)x^{n+k-2} = -x
\]  

(3.10)

Writing first few terms of the series on the left we obtain

\[
2\alpha a_2 + (2a_2 a_2 + 6\alpha a_2)x + (12\alpha a_4 + 6a_2 a_4 + 2a_2^2)x^2 + (20\alpha a_5 + 12a_2 a_4 + 8a_2 a_4)x^3 + ... = -x
\]  

(3.11)
from which we get

\[ a_2 = 0, \quad a_3 = -\frac{1}{6\alpha}, \quad a_4 = 0, \quad a_5 = 0 \]  

(3.12)

Define new summation index \( m = n + k - 2 \) and change the order of summation in (3.10). We get

\[
\sum_{m=0}^{\infty} \sum_{k=0}^{m} a_k a_{m-k+2} (m-k+2)(m-k+1) x^m = -x
\]

(3.13)

On the left side of (3.13) coefficient of each power of \( x \) other than unity must vanish. This gives for \( m \geq 2 \)

\[
\sum_{k=0}^{m} a_k a_{m-k+2} (m-k+2)(m-k+1) = 0
\]

(3.14)

which can be re-written as

\[
(m + 2)(m + 1)a_{m+2}a_0 = -\sum_{k=1}^{m} a_k a_{m-k+2} (m-k+2)(m-k+1)
\]

We have obtained the recurrence relation

\[
(m + 2)(m + 1)a_{m+2} = -\frac{1}{\alpha} \sum_{k=1}^{\infty} a_k a_{m-k+2} (m-k+2)(m-k+1), \quad m \geq 2
\]

(3.15)

If 3 is not a divisor of \( m + 2 \) it will not divide both \( k \) and \( m-k+2 \). From (3.9) and (3.12) we note that, with the exception of \( a_0 \) and \( a_3 \), other coefficients upto and including \( a_5 \) are all zero. Using this in (3.15) we find that \( a_6 \neq 0 \) but \( a_7 = a_8 = 0 \). Applying the principle of mathematical induction we can easily establish that only those coefficients whose suffixes are multiples of 3 fail to vanish. We can replace (3.15) by

\[
3n(3n-1)a_{3n} = -\frac{1}{\alpha} \sum_{k=1}^{n-1} (3n-3k)(3n-3k-1)a_{3k}a_{3n-3k}, \quad n \geq 2
\]

(3.16)

Define

\[
a_{3n} = -\frac{b_{3n}}{(3n)!\alpha^{2n-1}}
\]

(3.17)

then (3.16) becomes
\[3n(3n - 1) \frac{(-b_{3n})}{(3n)!\alpha^{2n-1}} = -\frac{1}{\alpha} \sum_{k=1}^{n-1} (3n - 3k)(3n - 3k - 1) \frac{(-b_{3k})}{(3k)!\alpha^{2k-1}} \frac{(-b_{3n-3k})}{(3n - 3k)!\alpha^{2(n-3k)-1}}\]

Or
\[b_{3n} = \sum_{k=1}^{n-1} \frac{(3n - 2)!}{(3k)!(3n - 3k - 2)!} b_{3k} b_{3n-3k}\]

Or
\[b_{3n} = \sum_{k=1}^{n-1} \frac{(3n - 2)}{3k} b_{3k} b_{3n-3k}\]  \hspace{1cm} (3.18)

From (3.17) we get
\[a_0 = -\frac{b_0}{\alpha} = -b_0\alpha\]
\[a_3 = -\frac{b_3}{6\alpha}\]

therefore  \(b_0 = -1, \ b_3 = 1\).

Formula (3.18) with \(b_0 = -1, \ b_3 = 1\) gives a solution of Wang's equation similar to Blasius solution found in the last section.

In (3.18) let \(n = 2\). We get
\[b_6 = \binom{4}{3} b_3 b_3 = 4\]

Also \(n = 3\) give,
\[b_9 = \sum_{k=1}^{2} \binom{7}{3k} b_{3k} b_{9-3k} = \binom{7}{3} b_3 b_6 + \binom{7}{6} b_6 b_3 = 168\]

Then (3.17) gives,
\[a_6 = -\frac{4}{6!\alpha^5} = -\frac{1}{180\alpha^5}\]
\[a_9 = -\frac{168}{9!\alpha^7} = -\frac{1}{2160\alpha^7}\]

We calculate first few coefficients in the above manner and list them in the following lines.
\begin{align*}
 b_0 &= -1, \quad a_0 = \alpha \\
 b_1 &= 1, \quad a_1 = -\frac{1}{6\alpha} \\
 b_2 &= 4, \quad a_2 = -\frac{1}{180\alpha^3} \\
 b_3 &= 168, \quad a_3 = -\frac{1}{2160\alpha^5} \\
 b_{12} &= 25200, \quad a_{12} = -\frac{1}{19008\alpha^7} \\
 b_{15} &= 9168432, \quad a_{15} = -\frac{7.01125 \times 10^{-6}}{\alpha^9} \\
 b_{18} &= 6594561792, \quad a_{18} = -\frac{1.03002 \times 10^{-6}}{\alpha^{11}} \\
 b_{21} &= 8257020125920, \quad a_{21} = -\frac{1.61614 \times 10^{-7}}{\alpha^{13}} \\
 b_{24} &= 16498091239838208, \quad a_{24} = -\frac{2.65906 \times 10^{-8}}{\alpha^{15}} \\
 b_{27} &= 49377815204079398400, \quad a_{27} = -\frac{4.53471 \times 10^{-9}}{\alpha^{17}} \\
 b_{30} &= 210964851697107227166720, \quad a_{30} = -\frac{7.95335 \times 10^{-10}}{\alpha^{19}} \\
 \vdots 
\end{align*}

Hashim [15] found the first nine coefficients (up to \(a_{24}\)) of (3.8), in complete agreement with members of the above list. Equation (3.16) can be easily used in a computer algorithm to extend the series (2.7) to an arbitrary number of terms with very little effort. As an example we give below coefficient of the \(x^{1500}\) term in the series (3.8).

\[ a_{1500} = -\frac{4.1809830 \times 10^{-335}}{\alpha^{999}} \]
3.4 An increasing sequence

To find \( \alpha \) we have to solve the equation \( y'(1) = 0 \), where

\[
y(x) = \sum_{n=0}^{\infty} a_{3n} x^{3n}
\]

This gives

\[
\alpha + \sum_{n=1}^{\infty} \frac{a_{3n}}{\alpha^{3n-1}} = 0
\]

Hashim [15] solved (3.20) after truncating the series after 7 terms and found \( \alpha \approx 0.463662 \). He improved this value to 0.466799 by approximating the series for \( y(x) \) up to \( x^{21} \) by the [12/12] Pade approximant. Since we can find an arbitrary number of terms in the series (3.20), we can attempt to solve (3.23) with a large number of terms. Let \( F(\alpha) \) denote the left side of Eq. (3.20) and let

\[
F_k(\alpha) = \alpha + \sum_{n=1}^{k} \frac{a_{3n}}{\alpha^{3n-1}} \quad k = 1, 2, 3, \ldots
\]

and let \( \alpha_k \) be a root of the equation

\[
F(\alpha) = 0
\]

We shall solve (3.8) by a slightly modified Newton's method. We first find \( \alpha_i \) by solving \( F_i(\alpha) = 0 \). To find \( \alpha_k , k \geq 2 \), we use the iterative scheme

\[
x_i = \alpha_{i-1}
\]

\[
x_{i+1} = x_i - \frac{F_k(x_i)}{F'_k(\alpha_{i-1})}, \quad i = 1, 2, \ldots
\]

and we approximate the derivative at \( \alpha_{i-1} \) by

\[
F'_k(\alpha_{i-1}) = \frac{F_k(\alpha_{i-1} + h) - F_k(\alpha_{i-1})}{h}
\]

and choose \( h \) a sufficiently small number. The above scheme works efficiently and produces a sequence some of whose members are given in the following table.
Table 4
Roots of Eq. (3.22)

The above sequence increases because for larger \( k \) more terms in the series (3.23) are retained and a larger value of \( \alpha \) will be needed to satisfy the equation. The sequence converges very slowly and no estimate about the accuracy of any term is available. In the next section we shall obtain a decreasing sequence also converging \( \alpha \). The two sequences together will determine precisely lower and upper bounds for \( \alpha \) at each level of approximation.

3.5 A decreasing sequence for \( \alpha \).

Hashim [15] and Wang [24] have found the following expression for \( \frac{1}{y} \)

\[
\frac{1}{y} = \frac{1}{\alpha} + \frac{x^3}{6\alpha^3} + \frac{x^6}{30\alpha^5} + \frac{x^9}{144\alpha^7} + \frac{2099x^{12}}{1425600\alpha^9} + \ldots
\] (3.25)

\[
= \frac{1}{\alpha} + \sum_{n=1}^{\infty} c_{3n} \frac{x^{3n}}{\alpha^{2n+1}}
\]

where \( c_{3n} \) is defined to be the coefficient of \( \frac{x^{3n}}{\alpha^{2n+1}} \) in (3.25). The solution \( y(x) \) of (3.7) is related to (3.25) in the following manner

\[
y(x) = \alpha - \int_{0}^{x} \frac{x}{y} \, dx
\] (3.26)

If we use (3.25) in (3.26), we obtain
\[ y(x) = -\frac{x^3}{6\alpha} - \sum_{n=1}^{\infty} c_{3n} \frac{x^{3n+3}}{(3n+3)(3n+2)\alpha^{2n+1}} \quad (3.27a) \]

\[ = -\sum_{n=0}^{\infty} \frac{c_{3n+3} x^{3n+3}}{(3n+3)(3n+2)\alpha^{2n+1}} \]

\[ = -\sum_{n=1}^{\infty} \frac{c_{3n-3} x^{3n}}{3n(3n-1)\alpha^{2n-1}} \quad (3.27b) \]

Compare (3.27.b) with (3.19), we find

\[ c_{3n-3} = -3n(3n-1)\alpha_{3n}, \quad n \geq 1 \quad (3.28) \]

Let \( x = 1 \) in (3.27.b). We shall truncate the series after \( k \) terms and find an approximate sum of the remainder. We have

\[ 0 = y(1) = -\sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha^{2n-1}} - \frac{c_{3k}}{(3k+3)(3k+2)\alpha^{2k+1}} - \ldots \quad (3.29) \]

If \( k \) is large, we can approximately set

\[ \frac{c_{3k+3}}{c_{3k}} = \alpha^2 \]

\[ \frac{c_{3k+6}}{c_{3k}} = \alpha^4 \]

and so on. Eq. (3.29) becomes

\[ 0 = -\sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha^{2n-1}} - \frac{c_{3k}}{(3k+3)(3k+2)\alpha^{2k+1}} - \sum_{n=k}^{\infty} \frac{1}{(3n+3)(3n+2)} \quad (3.30) \]

It is easy to show that

\[ \sum_{n=k}^{\infty} \frac{1}{3n+3} = \frac{1}{3k+2} \]

and (3.30) becomes

\[ 0 = -\sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha^{2n-1}} - \frac{c_{3k}}{(3k+2)\alpha^{2k+1}} - \frac{1}{3k+2} \]

In the notation of Section 3, the above equation may be written as

\[ F_k(\alpha) + \frac{3k+3}{\alpha^{2k+1}} a_{3k+3} = 0 \quad (3.31) \]

Since the approximation used to obtain (3.30) replaces a larger number by a smaller one, we have
\[ F(\alpha) \geq G(\alpha) \]

Where \( F(\alpha) \) and \( G(\alpha) \) denote respectively the left sides of Eqs. (3.20) and (3.31). The above inequality ensures that if \( \alpha_e \) is a zero of \( F(\alpha) \) then a zero of \( G(\alpha) \) greater than \( \alpha_e \) exists. We solve (5.3) for \( k=1,2,\ldots \) by the modified Newton's method.

Let \( \alpha^{(k)} \) denote a root of (5.3). We display a few members of this sequence in Table 5.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \alpha^{(k)} )</th>
<th>( k )</th>
<th>( \alpha^{(k)} )</th>
<th>( k )</th>
<th>( \alpha^{(k)} )</th>
<th>( k )</th>
<th>( \alpha^{(k)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.482463</td>
<td>100</td>
<td>0.470006</td>
<td>900</td>
<td>0.469633</td>
<td>2100</td>
<td>0.4696130</td>
</tr>
<tr>
<td>7</td>
<td>0.478413</td>
<td>200</td>
<td>0.469784</td>
<td>1000</td>
<td>0.4696298</td>
<td>3000</td>
<td>0.4696088</td>
</tr>
<tr>
<td>15</td>
<td>0.473268</td>
<td>300</td>
<td>0.469716</td>
<td>1500</td>
<td>0.4696190</td>
<td>3500</td>
<td>0.4696074</td>
</tr>
<tr>
<td>50</td>
<td>0.470505</td>
<td>500</td>
<td>0.469665</td>
<td>1800</td>
<td>0.4696155</td>
<td>4000</td>
<td>0.4696064</td>
</tr>
</tbody>
</table>

Table 5

Roots of Eq. (3.31)

Using the last entries of Tables 4 and 5 we have the following inequality for \( \alpha \)

\[ 0.469597 < \alpha < 0.4696064 \]

(3.32)

3.6 Pade approximation

In chapter 2 we discussed Hashim's results which were found by approximating the power series solution of Wang by a Pade approximant. Letting \( x=1 \) and putting the numerator equal to zero. From the [12/12] approximant he found \( \alpha = 0.466799 \).

We have shown above that the exact value of \( \alpha \) lies between 0.4695975 and 0.4696064. Thus Hashim's result is correct to only two decimal positions and, in order to achieve a more accurate result by this method, Pade' approximants of very high order must be used. However the amount of work involved in ADM becomes prohibitive after a few terms which
precludes the use of this approach to high orders. We can use our solution to carry this method to higher orders to generate a sequence of numbers which should converge to $\alpha$. We find that, up to a point, this sequence increases and appears to be converging to the exact value of $\alpha$, each term being a better approximation as compared with the corresponding number given by a straightforward solution of the equation $y(1)=0$ after truncating the series (2.7) and retaining an appropriate number of terms. However after this stage the Pade' approximations start yielding numbers which, instead of giving an improved approximation, produce errors of increasing magnitude. Thus this method appears to be of limited utility.

We illustrate our method by an example. Let us truncate the series (2.7) after five terms and let $x=1$. We get an approximate value of $\alpha$ by solving the equation
\[
\alpha - \frac{1}{6\alpha} - \frac{1}{180\alpha^3} - \frac{1}{2160\alpha^5} - \frac{1}{19008\alpha^7} = 0
\]  
(3.33)

If we divide through with $\alpha$ and let $p = \frac{1}{\alpha^2}$, the equation becomes
\[
1 - \frac{p}{6} - \frac{p^2}{180} - \frac{p^3}{2160} - \frac{p^4}{19008} = 0
\]  
(3.34)

We replace the left side of (3.34) by its [2/2] Padé approximant. A Padé approximant can be easily found by using algebraic softwares like MATLAB or Mathematica. This gives
\[
1 - \frac{119}{396}p + \frac{73}{3960}p^2 - \frac{53}{396}p + \frac{1}{594}p^2 = 0
\]  
(3.35)

By equating the numerator of the fraction on the left side of (3.35) to zero we find $p = 4.65968$. An approximate value of $\alpha$ is given by $\frac{1}{\sqrt{p}} = 0.463257$. This agrees with the entry which appears in the table of...
Hashim[15] corresponding to his Pade [6/6] approximant. In Table 6 we present approximate values of $\alpha$ calculated by using successive Pade' approximants starting with the [2/2] approximant and going up to the [23/23] approximant. For comparison, we also include values of this parameter obtained by solving an equation of the from (3.34) after steadily increasing the number of terms. Note that our [23/23] approximant corresponds to the [69/69] approximant of Hashim assuming that one could go as far as that by adopting the ADM approach.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$\alpha^i$</th>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$\alpha^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.457674</td>
<td>0.463257</td>
<td>14</td>
<td>0.468525</td>
<td>0.469075</td>
</tr>
<tr>
<td>3</td>
<td>0.462404</td>
<td>0.466791</td>
<td>15</td>
<td>0.468611</td>
<td>0.469086</td>
</tr>
<tr>
<td>4</td>
<td>0.464572</td>
<td>0.468061</td>
<td>16</td>
<td>0.468686</td>
<td>0.469097</td>
</tr>
<tr>
<td>5</td>
<td>0.465791</td>
<td>0.468645</td>
<td>17</td>
<td>0.468750</td>
<td>0.469107</td>
</tr>
<tr>
<td>6</td>
<td>0.466561</td>
<td>0.468956</td>
<td>18</td>
<td>0.468807</td>
<td>0.469118</td>
</tr>
<tr>
<td>7</td>
<td>0.467088</td>
<td>0.468980</td>
<td>19</td>
<td>0.468857</td>
<td>0.469123</td>
</tr>
<tr>
<td>8</td>
<td>0.467469</td>
<td>0.468997</td>
<td>20</td>
<td>0.468901</td>
<td>0.469124</td>
</tr>
<tr>
<td>9</td>
<td>0.467756</td>
<td>0.469011</td>
<td>21</td>
<td>0.468941</td>
<td>0.469053</td>
</tr>
<tr>
<td>10</td>
<td>0.467980</td>
<td>0.469025</td>
<td>22</td>
<td>0.468977</td>
<td>0.469977</td>
</tr>
<tr>
<td>11</td>
<td>0.468158</td>
<td>0.469038</td>
<td>23</td>
<td>0.469009</td>
<td>0.474672</td>
</tr>
<tr>
<td>12</td>
<td>0.468303</td>
<td>0.469051</td>
<td>24</td>
<td>0.469039</td>
<td>no value</td>
</tr>
<tr>
<td>13</td>
<td>0.468424</td>
<td>0.469063</td>
<td>25</td>
<td>0.469066</td>
<td>no value</td>
</tr>
</tbody>
</table>

Table 6. Values of $\alpha_i$ found by solving the equation $y(1)=0$, when we retain $2i+1$ terms in (2.7) and $\alpha^i$ given by the $[i/i]$ Pade' approximant.

A glance at the Table indicates that successive terms of the sequence $\alpha^i$ give better approximation to the exact value of $\alpha$ for $i =2,3,\ldots,22$ but $\alpha^{23}$ is much worse compared with $\alpha_{23}$. Also the sequence ends abruptly at $\alpha^{23}$ because the Pade' [24/24] approximant equated to zero does not yield a real root leading to a suitable entry in the Table. We are faced with a dilemma. On the one hand the sequence $\alpha_i$ continues beyond $i =23$ and the
subsequent terms yield an increasing sequence which converges to $\alpha$, on the other hand the corresponding Pade' approximant either produces an $\alpha'$ which is worthless or it fails to generate any value of $\alpha'$ at all. This is strange since it is well known that an \([m/m]\) Pade' approximant gives a better representation of a function than its Taylor series truncated after 2m terms. The first 22 terms of Table 4 mirror this fact. The anomalous behavior after $i=23$ can be explained in only one way that the Pade' approximants from the stage \([23/23]\) onwards no longer adequately represent the corresponding Taylor polynomial on an interval containing the unique zero of the polynomial. Probably this occurs due to ill-conditioned matrices which are involved in the calculation of coefficients in the Pade' approximants of sufficiently high orders.

3.7 The integral $\int_0^\infty [f'(\eta)]^2 d\eta$.

Integrate (2.7) with respect to $x$ on \([0,1)\). We get

$$\int_0^1 ydx = \alpha - \frac{1}{4.6\alpha} - \frac{1}{7.180\alpha^3} - \frac{1}{10.2160\alpha^5} \ldots \quad (3.36)$$

$$< \alpha \quad (3.37)$$

Also we have from (3.36)

$$\int_0^1 ydx > \alpha - \frac{1}{4} \left( \frac{1}{6\alpha} + \frac{1}{180\alpha^3} + \frac{1}{2160\alpha^5} + \ldots \right)$$

$$= \alpha - \frac{1}{4} \alpha \quad (3.38)$$

$$= \frac{3}{4} \alpha \quad (3.39)$$

Eq. (3.38) follows from the line preceding it because $\alpha$ satisfies the equation $y(1)=0$, which becomes, on letting $x=1$ in (2.7),

$$\alpha = \frac{1}{6\alpha} + \frac{1}{180\alpha^3} + \ldots$$
Now \( y = \frac{d^2f}{d\eta^2} \) and \( x = \frac{df}{d\eta} \). Hence

\[
\int_{0}^{1} y \, dx = \int_{0}^{\infty} \left( \frac{d^2f}{d\eta^2} \right)^2 \, d\eta
\]

(3.40)

From (3.37), (3.39) and (3.40) we obtain the inequality,

\[
\frac{3}{4} \alpha < \int_{0}^{\infty} \left( \frac{d^2f}{d\eta^2} \right)^2 \, d\eta < \alpha
\]

(3.41)

If we use \( \alpha = 0.469606 \) in (3.36) then we can replace (3.41) by the equality

\[
\int_{0}^{\infty} \left( \frac{d^2f}{d\eta^2} \right)^2 \, d\eta = 0.37118
\]

(3.42)

Write the Blasius equation \( f^{\prime\prime\prime\prime} + f f^{\prime\prime} = 0 \) in the form

\[
\frac{f^{\prime\prime\prime\prime}}{f^\prime} = -f
\]

An integration gives

\[
f^{\prime\prime\prime}(\eta) = \alpha \exp \left( -\int_{0}^{\eta} f(u) \, du \right)
\]

(3.43)

It is clear from (3.43) that for \( 0 \leq \eta < \infty \), \( f^{\prime\prime\prime}(\eta) \geq 0 \). As a consequence \( f^{\prime}(\eta) \) is nonnegative and monotonically increasing and \( f(\eta) \) is also increasing and unbounded. Since \( f^{\prime}(\eta) \) is bounded above by unity, the curve \( y = f(\eta) \) does not intersect the line \( y = \eta \), because otherwise \( f(\eta) \) being a convex function, its slope, at the point of intersection must exceed unity. This means the graph of \( y = f(\eta) \) is flanked on the left by the line \( y = \eta \). The initial condition \( \lim_{\eta \to \infty} f^{\prime}(\eta) = 1 \) indicates that the function \( f(\eta) \) has an asymptote with slope unity but lying below the line \( y = \eta \). Let the equation of this asymptote be

\[
y = \eta - \eta_0
\]

Since \( \int_{0}^{\eta} f(\eta) \, d\eta \geq \int_{\eta_0}^{\eta} (\eta - \eta_0) \, d\eta \), an exponentiation leads to the inequality
\[ f''(\eta) = \alpha \exp \left( - \frac{1}{2} (\eta - \eta_0)^2 \right) \]

Square both sides of (3.44) and integrate on [0, \infty). We find

\[ 0.37118 \leq \alpha^2 \int_0^\infty \exp \{-(\eta - \eta_0)^2\} \]

where we have used (3.42). Now

\[
\int_0^\infty \exp \{-(\eta - \eta_0)^2\} = \int_0^{\eta_0} e^{-u^2} du + \int_0^{\infty} e^{-u^2} du
\]

\[ = \frac{\sqrt{\pi}}{2} + \frac{\eta_0}{2} \]

Combining (3.45) and (3.46) we obtain

\[ \int_0^{\eta_0} e^{-u^2} du \geq \frac{0.37118 \alpha^2}{2} - \frac{\sqrt{\pi}}{2} \]

\[ = 0.79695 \]

The above inequality is satisfied if

\[ \eta_0 \geq 1.16 \]

The above result indicates that the line

\[ y = \eta - 1.16 \]

is approximately an asymptote to the curve \( y = f(\eta) \). Since this curve eventually behaves like a straight line of slope unity, it follows that, for large \( \eta \),

\[ f(\eta) \approx \eta - 1.16 \]

A numerical solution of the Blasius problem indicates that \( \eta_0 = 1.23 \). The above result gives this number with an error less than 6 percent.

We were able to apply a classical method to solve Wang's equation which gave the solution to an arbitrary number of terms. In principle a modern
method such as the Adomian decomposition method also gives the solution to as many terms as we wish but the computational effort involved to calculate the 3000th term, for example, would be tremendous. However our advantage in easy calculation of the two sequences \( \{\alpha_k\} \) and \( \{\alpha^{(k)}\} \), is somewhat neutralized by their extremely slow convergence to \( \alpha \).
Chapter 4 Limitations of Adomian Decomposition Method

4.1 Introduction

In the last two chapters we solved Blasius equation and its transformed version, Wang equation, by Adomian decomposition method as well as by a direct method. In both cases the solution comes out in the form of an infinite series. However the series converges within a finite interval and it is impossible to get any information about the solution for large values of $\eta$. To evaluate $f''(0)$, correct to six decimal places, we need to find a series solution of the Wang equation up to several thousand terms. Even after using this value of $f''(0)$ to solve the Blasius problem, the series solution converges only for $\eta \leq 5.90$. To get a complete picture we have to solve the Blasius problem numerically such as Runge-Kutta method.

Liao has developed a method, called the homotopy analysis method, which attempts to accelerate the converge of the series solution. In chapter 1 we described his method of solution. However there are a number of arbitrary parameters in the method and one has to depend on guess work to estimate a suitable value of them.

In general it is useful to make an asymptotic analysis of a problem before tackling it by the Adomian or the homotopy analysis methods. Such methods produce series solutions which, for nonlinear problems, only converge within a small interval.
4.2 A simple problem

To highlight the deficiencies of the Adomian and the homotopy analysis method, we consider a very simple nonlinear problem

\[
\frac{d^2 y}{dx^2} + y^2 = x \tag{4.50}
\]
\[y(0) = 0, \quad y'(0) = 0 \tag{4.51}\]

First we find an approximate analytical solution for large \(x\). Consider a transformation

\[y = u + \sqrt{x} \tag{4.52}\]

Then

\[y' = u' + \frac{1}{2} x^{-\frac{1}{2}} \]
\[y'' = u'' - \frac{1}{4} x^{-\frac{3}{2}} \tag{4.53}\]

and \[y^2 = u^2 + x + 2u\sqrt{x} \tag{4.54}\]

For \(x \gg 1\), the second term on the right of (4.53) may be dropped. Also we assume \(|u| << \sqrt{x}\), an assumption justified later. Therefore we drop \(u^2\) in (4.54) and Eq. (4.50) is transformed to a linear equation

\[u'' + 2u' \sqrt{x} = 0 \tag{4.55}\]

Now it is well-known that the equation

\[x^2 \frac{d^2 y}{dx^2} + (1 - 2s)x \frac{dy}{dx} + [(s^2 - r^2 + \alpha^2) \alpha^2 + a^2 r^2 x^{2r}] y = 0 \tag{4.56}\]

has the general solution

\[y = x^s [c_1 J_{\alpha}(ax') + c_2 Y_{\alpha}(ax')] \tag{4.57}\]

Multiply (4.55) with \(x^2\). We get

\[x^2 u'' + 2x \frac{5}{2} u = 0 \tag{4.58}\]
Compare (4.58) with (4.56). We get
\[
\begin{align*}
1 - 2s &= 0 \\
s^2 - r^2\alpha^2 &= 0 \\
2r &= \frac{5}{2} \\
a^2r^2 &= 2
\end{align*}
\]

Therefore \( s = \frac{1}{2}, \ r = \frac{5}{4}, \ \alpha = \frac{2}{5}, \ a = \frac{4\sqrt{2}}{5} \) and the general solution of (4.55) is

\[
u = \sqrt{x} \left[ c_1 J_{\frac{5}{2}} \left( \frac{4\sqrt{2}}{5} x^{\frac{5}{4}} \right) + c_2 Y_{\frac{5}{2}} \left( \frac{4\sqrt{2}}{5} x^{\frac{5}{4}} \right) \right]
\]

Since the Bessel functions are oscillatory with decreasing amplitude such that

\[
J_{\alpha}(x) \to 0 \quad \text{as} \quad x \to \infty \\
\text{and} \quad Y_{\alpha}(x) \to 0 \quad \text{as} \quad x \to \infty
\]

our assumption that \(|\nu| << \sqrt{x}\) is justified for large \(x\).

From (4.52) we see that the solution of Eq. (4.50), for large \(x\), will be small oscillations superimposed on the parabola \(y = \sqrt{x}\).

In Fig.4 we present the numerical solution of the problem which confirms our qualitative analysis.
Fig. 4: Numerical solution of Eq. (4.50).

In Fig. 5 we present the solution along with the curve \( y = \sqrt{x} \), shown dashed.

Fig. 5: The solution of Eq. (4.50) together with \( y = \sqrt{x} \) shown dashed.
4.3 Solution by Adomian method

If we solve the problem by Adomian decomposition method, we get the following solution

\[
y = \frac{x^3}{6} - \frac{x^8}{2016} + \frac{x^{13}}{943488} - \frac{95x^{18}}{48502831104} + \frac{31x^{23}}{9203412201984} - \frac{13507369511417413632}{74849x^{28}} + \frac{213956733060851831903088}{189251x^{33}} \\
- \frac{38700523x^{38}}{2801579230592089310166515712} + \frac{1061168785x^{43}}{50011106613036187601787024310272} \\
- \frac{47494608245909822416547577359008530432}{1527994261379x^{48}} + \frac{986872159994333x^{53}}{20444184729624559311214550180022959197913088} \\
- \frac{3659317869389202813517x^{58}}{5096028520373191763798652378101228449208971561009152} \\
+ \frac{63328266525722085749107x^{63}}{597156645001733061088192608566590194967830728751905243136}
\]

In Fig. 6 we plot the above solution. It is clear that the series diverges when \( x > 3.5 \) and the oscillatory behavior depicted in Figs. 4 and 5 is nowhere to be seen.
If we add one more term to the solution it becomes

\[
y = 0.166667x^3 - 0.000496032x^8 + 1.0599 \times 10^{-6}x^{13} - 1.95865 \times 10^{-9}x^{18} + 3.36832 \times 10^{-12}x^{23} - 5.54135 \times 10^{-15}x^{28} + 8.84529 \times 10^{-18}x^{33} - 1.38138 \times 10^{-20}x^{38} + 2.12187 \times 10^{-23}x^{43} - 3.2172 \times 10^{-26}x^{48} + 4.82715 \times 10^{-29}x^{53} - 7.18068 \times 10^{-32}x^{58} + 1.0605 \times 10^{-34}x^{63} - 1.55666 \times 10^{-37}x^{68}.
\]

The graph of shown in Fig. 7. Again the divergence of the series is clear beyond \( x = 3.3 \).
4.4 Homotopy Analysis Method

It has been shown by Liao [19] that the Adomian solution up to m terms

\[ y = \sum_{n=0}^{m} a_n x^n \]

Can be improved by the homotopy analysis method to

\[ y = \sum_{n=0}^{m} \mu_{m,n}(h) a_n x^n \]  (4.59)

where \( \mu_{m,n}(h) \) is called an approaching function and is defined as

\[ \mu_{m,n}(h) = (-h)^n \sum_{k=0}^{m+n} \left( \frac{n-1+k}{k} \right) (1+h)^k \]  (4.60)

where \( h \) is arbitrary and should be chosen so that the solution (4.59) agrees with the exact solution over as wide an interval as possible.

If we choose \( m = 7 \), the homotopy solution becomes

\[
\begin{align*}
    y &= \frac{1}{6} (1-h)(1+h+h^2+h^3+h^4+h^5+h^6)x^3 \\
    &\quad - \frac{(1-h)^2(1+2h+3h^2+4h^3+5h^4+6h^5)x^8}{2016} + \frac{(1-h)^3(1+3h+6h^2+10h^3+15h^4)x^{13}}{943488} \\
    &\quad - \frac{95(1-h)^4(1+4h+10h^2+20h^3)x^{18}}{48502831104} + \frac{31(1-h)^5(1+5h+15h^2)x^{23}}{9203412201984} \\
    &\quad - \frac{74849(1-h)^6(1+5h+15h^2)x^{28}}{13507369511417413632} + \frac{189251(1-h)^7(1+6h)x^{33}}{21395673306085183193088}
\end{align*}
\]

We plot the above solution for various values of \( h \) and observe that the homotopy solution also suffers from the convergence problem. In any case it fails to produce the oscillatory behavior of the solution shown in Fig. 4 and 5.
Fig. 8: Homotopy solution with $h = 0.1$

Fig. 9: Homotopy solution with $h = -0.1$
Fig. 10: Homotopy solution with $h = 0.2$

Fig. 11: Homotopy solution with $h = -0.2$
We note that the homotopy solution in Figs. 11-13 produces a zero but the exact solution is free of zeros. This shows that a bad choice of the parameter $h$ can lead to a solution which is totally wrong even for small values of $x$. 
4.5 Conclusion

By applying the Adomian and homotopy analysis method to our simple nonlinear problem we have shown that these methods produce series solutions which converge over a relatively short interval and it is impossible to get any information for large $x$. For this one should turn to asymptotic analysis of the problem.
References