Question 1:

(a) Let \( f : [0, 2] \to \mathbb{R} \) be defined by \( f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ x, & 1 < x \leq 2 \end{cases} \).

Show by definition that \( f \) is Riemann integrable on \([0, 2]\).

(b) Let \( f : [a, b] \to \mathbb{R} \) be integrable on \([a, b]\) and let \( F : [a, b] \to \mathbb{R} \) be a continuous and differentiable function on \([a, b]\) with \( F'(x) = f(x), \forall x \in (a, b) \). Show that

\[
\int_a^b f = F(b) - F(a).
\]

(c) Suppose that \( F : [0, \infty) \to \mathbb{R} \) is continuous and that \( f(x) \neq 0, \forall x > 0 \). If we have

\[
(f(x))^2 = 2\int_0^x f, \quad \forall x > 0, \text{ show that } f(x) = x, \forall x \geq 0.
\]

Question 2:

Prove or disprove the following:

(a) \( f \) is Riemann integrable on \([a, b]\) if and only if \( |f| \) is Riemann integrable on \([a, b]\).

(b) Given any partitions \( P, Q \) of \([a, b]\), for any function \( f : [a, b] \to \mathbb{R} \), we have

\[
L(f, P) \leq U(f, Q).
\]

(c) Any integrable function on \([a, b]\) has an antiderivative.
Question 1:

(a) Let \( f_n : [a, b] \to R \) be a sequence of integrable functions on \([a, b] \). If \( \sum f_n \) converges uniformly on \([a, b] \) to a function \( f \), show that \( f \) is integrable on \([a, b] \) and \( \int_a^b f = \sum \int_a^b f_n \).

(b) Evaluate \( \int_1^2 \sum_{n=1}^\infty \frac{x^{n-1}}{n^2 (1 + x^n)} \, dx \).

Question 2:

(a) Show that if \( \{ E_i : i \in N \} \) is a family of disjoint measurable subsets of \( R \), then \( m(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m(E_i) \).

(b) Prove that \( \forall E, F \in \mathcal{M}, m(E \cup F) = m(E) + m(F) - m(E \cap F) \). Then use it to show that if \( A, B \subseteq [0,1] \) with \( m(A) = 1 \), then \( m(B) = m(A \cup B) \).

(c) Show that \( m^* \{ (a,b) \} = b - a \).

Question 3:

Let \( f_n(x) = \frac{x^2 + nx}{n}, x \in R \)

(a) Find \( \lim_{n \to \infty} f_n(x) \).

(b) Show that \( (f_n) \) converges uniformly on \([0,8] \) to \( f \). What type is the convergence on \( R \).

(c) What is \( \lim_{n \to \infty} \int_1^4 f_n(x) \, dx \).

Question 4: Prove or disprove the following

1) If \( E \subseteq R \) and \( E \) is measurable, then so is \( -E \).

2) The union of two \( \sigma \)-algebras on a set is again a \( \sigma \)-algebra on the same set.

3) If \( (f_n) \) is a sequence of bounded functions on \([a,b] \) that converges uniformly on \([a,b] \) to \( f \), then \( f \) is also bounded on \([a,b] \).
Question 1:

(a) Show that if \( f : [a, b] \to \mathbb{R} \) is an increasing function, then it's Riemann integrable on \([a,b]\).

(b) Let \( I = [a, b] \) and let \( f, g \) be continuous on \( I \) and such that \( \int_a^b f = \int_a^b g \). Prove that there exists \( c \in I \) such that \( f(c) = g(c) \).

(c) Let \( I = [a, b] \) and let \( f, g, h \) bounded functions on \( I \) such that \( f(x) \leq g(x) \leq h(x) \) on \( I \).

If \( f, h \in R(a,b) \) and \( \int_a^b f = \int_a^b h \). Prove that \( g \in R(a,b) \) and \( \int_a^b f = \int_a^b g \).

Question 2:

(a) Let \( f_n(x) = x^n / n \) for \( x \in [0,1] \). Show that the sequence \( (f_n) \) of differentiable functions converges uniformly to a differentiable function \( f \) on \([0,1]\), and that the sequence \( (f'_n) \) converges on \([0,1]\) to a function \( g \), but \( g(1) \neq f'(1) \).

(b) Let \( (M_n) \) be a sequence of positive real numbers such that \( |f_n(x)| \leq M_n \) for \( x \in D \) and \( n \in N \). If \( \sum (M_n) \) is convergent, prove that \( \sum (f_n) \) is uniformly convergent on \( D \).

(c) Test \( \sum_{n=1}^{\infty} \left( \frac{\sin nx}{n} \right)^3 \) for uniform convergence.

Question 3:

(a) Let \( \Omega \in \mathcal{A} \). If the functions \( f, g : \Omega \to \mathbb{R} \) with \( f = g \) (a.e.) and the function \( f \) is measurable. Show that \( g \) is also measurable.

(b) Show that if \( f \) is Lebesgue integrable on \( \Omega \) then so is \( |f| \), and \( \left| \int_\Omega f \ dm \right| \leq \int_\Omega |f| \ dm \).

(c) If \( E \subseteq \mathbb{R} \), show that \( \forall \varepsilon > 0, \exists G \) open and \( E \subseteq G \) and \( m^*(G) \leq m^*(E) + \varepsilon \).
**Question 4:**

(a) State Fatou's Lemma and use it to prove the Dominated Convergence Theorem.

(b) Evaluate \( \lim_{n \to \infty} \int_0^1 \frac{nx \log x}{1+(nx)^2} \, dx \).

**Question 5:**

(a) Show that if \( f \geq 0 \) on \( [a, \infty) \) and \( \int_a^\infty f \) converges then \( f \in L^1([a, \infty)) \) and \( \int_a^\infty f = \int_{[a, \infty)} f \).

(b) Explain by examples two properties of Lebesgue integral which are not satisfy by Riemann integral.

**Question 6: Prove or disprove the following:**

(1) If \( \Omega_1, \Omega_2 \subseteq \Omega \) are measurable subsets of \( \Omega \) and \( f|_{\Omega_1} \) and \( f|_{\Omega_2} \) are measurable functions then \( f \) is measurable on \( \Omega_1 \cup \Omega_2 \).

(2) The composition of two Riemann integrable functions is Riemann integrable.

(3) If \( A, B \) are measurable subsets of \( \mathbb{R} \), then \( A \Delta B \) is also measurable.

(4) If \( h:[0,1] \to \mathbb{R} \) is Riemann integrable with \( h(x) > 0 \) for all \( x \), then \( 1/h \) is Riemann integrable.

(5) If \( \chi_E \) is measurable function then \( E \) is a measurable set.