Mean-Value Theorem for Integrals: If \( f \) is a continuous function on the closed interval \([a, b]\), then there is at least one number \( c \), \( a \leq c \leq b \), so that

\[
f(c)(b - a) = \int_a^b f(x) \, dx.
\]

**Proof:** Since \( f \) is a continuous function on the closed interval \([a, b]\), by the Maximum- and Minimum-Value Theorems (pp. 79-80), \( f \) has a maximum value \( M \) and a minimum value \( m \) on \([a, b]\), i.e., \( m \leq f(x) \leq M \) on \([a, b]\). By property 7.) (p. 291) of definite integrals,

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a),
\]

so that

\[
m \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq M,
\]

and call this number \( y_o \).

By the Intermediate-Value Theorem (p. 81) there is at least one number \( c \), \( a \leq c \leq b \), so that

\[
f(c) = y_o, \text{ i.e., } f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx,
\]

so that

\[
f(c)(b - a) = \int_a^b f(x) \, dx.
\]

**First Fundamental Theorem of Calculus (FTC1):** Assume that \( f \) is a continuous function on the closed interval \([a, b]\) and that \( F(x) = \int_a^x f(t) \, dt \). Then \( F'(x) = f(x) \).

**Proof:** Consider \( F(x) = \int_a^x f(t) \, dt \) as the area under the graph of \( f \) above the interval \([a, x]\). Then \( F(x + h) \) is the area under the graph of \( f \) above the interval \([a, x + h]\) and \( F(x + h) - F(x) \) is the area of the "thin strip" from \( x \) to \( x + h \), i.e., \( F(x + h) - F(x) = \int_x^{x+h} f(t) \, dt \). By the Mean-Value Theorem for integrals there is at least one number \( c \), \( x \leq c \leq x + h \), so that

\[
f(c) \cdot h = \int_x^{x+h} f(t) \, dt.
\]
The derivative of $F(x)$ can now be computed as

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \int_x^{x+h} f(t) \, dt$$

$$= \lim_{h \to 0} \frac{f(c) \, h}{h}$$

$$= \lim_{h \to 0} f(c) \quad \text{(Recall that } x \leq c \leq x + h \text{.)}$$

$$= f(x).$$

**Second Fundamental Theorem of Calculus (FTC2):** Let $f$ be a continuous function on the closed interval $[a, b]$. Assume that $F(x)$ is an antiderivative of $f(x)$, i.e., assume that $F'(x) = f(x)$. Then

$$\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a).$$

**Proof:** Let $A(x) = \int_a^x f(t) \, dt \,$. Then $A(a) = 0$, $A(b) = \int_a^b f(t) \, dt$, and $A'(x) = f(x)$ by FTC1. But $F'(x) = f(x)$. By Corollary 2 (p. 168) to the Mean-Value Theorem $F(x) = A(x) + C$ for any constant $C$, or

$$A(x) = F(x) - C.$$ 

Then

$$\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$$

$$= A(b)$$

$$= A(b) - A(a)$$

$$= (F(b) - C) - (F(a) - C)$$

$$= F(b) - F(a)$$

$$= F(x) \bigg|_a^b.$$