6.3 Failure Theories

- There is no universal theory of failure for the general case of material properties and stress state.
- Several hypotheses have been formulated and tested, leading to today’s accepted theories for most designers.
- Structural metal behavior is typically classified as being ductile or brittle.
- Ductile material is a material that is characterized by its ability to yield at normal temperature. Example (Steel, Al, many other alloys of other metals).

\[
\text{If } \varepsilon_f \geq 0.05, \text{ the material is ductile, and it has an identifiable yield strength of } S_y = S_{yc} = S_y.
\]
- If $\varepsilon_f < 0.05$, the material is brittle, and it is classified by $S_{ut}$ and $S_{uc}$ (which is given in positive quantity)

- A flowchart for the famous failure theories that are used today is shown below.

I Ductile Materials

6.4 The Maximum Shear Stress Theory

- For the general state of stress, three principal stresses can be determine and ordered such that:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \quad (1)$$

- Then, the yielding will occur whenever:

$$\tau_{\text{max}} \geq \frac{S_y}{2} \quad \text{or} \quad \sigma_1 - \sigma_3 \geq S_y \quad (2)$$
Hence, for design purpose, equation (1) can be modified to incorporate a factor of safety $n$. Thus:

$$\sigma_1 - \sigma_3 = \frac{S_y}{n}$$  \hspace{1cm} (3)

For plane stress problem, it is common that one of the principle stresses is zero, and the other two $\sigma_A$ and $\sigma_B$ are determined from the principle equation.

Assuming $\sigma_A \geq \sigma_B$, there are three cases:

- **Case 1**: $\sigma_A \geq \sigma_B \geq 0$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$, equation (3) reduce to:
  $$\sigma_A = \frac{S_y}{n}$$  \hspace{1cm} (4)

- **Case(2)**: $\sigma_A \geq 0 \geq \sigma_B$, then $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$, equation (3) reduce to:
  $$\sigma_A - \sigma_B = \frac{S_y}{n}$$  \hspace{1cm} (5)

- **Case(3)**: $0 \geq \sigma_A \geq \sigma_B$, then $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$, equation (3) reduce to:
  $$\sigma_B = -\frac{S_y}{n}$$  \hspace{1cm} (6)
6.5 Distortion-Energy Theory for Ductile Material

- It is also called Von Mises Theory or Shear Energy Theory
- For 3-D: Von Mises theory can be defined in terms of xyz coordinate system as:

\[ \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yx}^2) \right]^{1/2} = \frac{S_y}{n} \]  (7)

When the equation defined in terms of the principal stresses:

\[ \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_2)^2 \right]^{1/2} = \frac{S_y}{n} \]  (8)

- For 2-D: Von Mises theory can be defined in terms of xy coordinate system as:

\[ \left[ \sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 \right]^{1/2} = \frac{S_y}{n} \]  (9)

When the equation defined in terms of the principal stresses:

\[ \left[ \sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2 \right]^{1/2} = \frac{S_y}{n} \]  (10)
6.6 Coulomb-Mohr Theory for Ductile Material (Internal Friction Theory)

- \( S_{yt} \neq S_{yc} \)
- Consider the conventional ordering of the principal stress such that \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \), then:
  \[
  \frac{\sigma_1 - \sigma_3}{S_{yt} - S_{yc}} = \frac{1}{n} \tag{11}
  \]
- In this theory either yield strength or ultimate strength can be used.
- For plane stress, when two nonzero principle stresses are \( \sigma_A \geq \sigma_B \), we have the following three cases:
  - Case 1: \( \sigma_A \geq \sigma_B \geq 0 \), then \( \sigma_1=\sigma_A \) and \( \sigma_3=0 \) and equation (11) reduce to:
    \[
    \sigma_A = \frac{S_{yt}}{n} \tag{12}
    \]
  - Case 2: \( 0 \geq \sigma_A \geq \sigma_B \), then \( \sigma_1=\sigma_A \) and \( \sigma_3=\sigma_B \) and equation (11) reduce to:
    \[
    \frac{\sigma_A - \sigma_B}{S_{yt} - S_{yc}} = \frac{1}{n} \tag{13}
    \]
  - Case 3: \( 0 \geq \sigma_A \geq \sigma_B \), then \( \sigma_1=0 \) and \( \sigma_3=\sigma_B \) and equation (11) reduce to:
    \[
    \sigma_B = -\frac{S_{yc}}{n} \tag{14}
    \]
- For pure shear \( \tau \), \( \sigma_1=\sigma_3=\tau \) and \( \tau_{max}=\tau_{sy} \), then equation (11) becomes:
  \[
  S_{sy} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}} \tag{15}
  \]
II Brittle Materials: (page 14-16 for reading only)

6.8 Maximum-Normal-Stress Theory for Brittle Materials (MNS)

- For the general state of stress, three principal stresses can be determined and ordered such that:
  \[ \sigma_1 \geq \sigma_2 \geq \sigma_3 \]  
  (16)

- Then, the failure occurs whenever:
  \[ \sigma_1 \geq S_{ut} \text{ or } \sigma_2 \leq -S_{uc} \]  
  (17)

- For plane stress problem, it is common that one of the principle stresses is zero, and the other two \( \sigma_A \) and \( \sigma_B \) are determined from the principle equation. Assuming \( \sigma_A \geq \sigma_B \), equation (17) can be written as:
  \[ \sigma_A \geq S_{ut} \text{ or } \sigma_B \leq -S_{uc} \]  
  (18)

- Converted equation (18) to design equation and considering four conditions, we have the following cases:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Equation to be used</th>
<th>Location of load line shown in Fig</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \sigma_A \geq \sigma_B \geq 0 )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} )</td>
<td>Load Line 1</td>
</tr>
<tr>
<td>2 ( \sigma_A \geq 0 \geq \sigma_B ) and ( \frac{\sigma_B}{\sigma_A} \leq \frac{S_{uc}}{S_{ut}} )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} )</td>
<td>Load Line 2</td>
</tr>
<tr>
<td>3 ( \sigma_A \geq 0 \geq \sigma_B ) and ( \frac{\sigma_B}{\sigma_A} &gt; \frac{S_{uc}}{S_{ut}} )</td>
<td>( \sigma_B = -\frac{S_{uc}}{n} )</td>
<td>Load Line 3</td>
</tr>
<tr>
<td>4 ( 0 \geq \sigma_A \geq \sigma_B )</td>
<td>( \sigma_B = -\frac{S_{uc}}{n} )</td>
<td>Load Line 4</td>
</tr>
</tbody>
</table>
6.9 Modifications of the Mohr Theory for Brittle Materials

- There are three modifications for the Mohr theory for brittle materials. The equations provided for each theory will be restricted to plane stress.

1- Brittle-Coulomb-Mohr

- For plane stress and with two nonzero principle stresses in which \( \sigma_A \geq \sigma_B \), we have the following three cases:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Equation to be used</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_A \geq \sigma_B \geq 0 )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} )</td>
</tr>
<tr>
<td>( \sigma_A \geq 0 \geq \sigma_B )</td>
<td>( \frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} )</td>
</tr>
<tr>
<td>( 0 \geq \sigma_A \geq \sigma_B )</td>
<td>( \sigma_B = -\frac{S_{uc}}{n} )</td>
</tr>
</tbody>
</table>

2- Modified I-Mohr

- On the basis of observed data for the fourth quadrant, the modified I-Mohr theory expands the fourth quadrant as shown in figure below.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Equation to be used</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_A \geq \sigma_B \geq 0 )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} )</td>
</tr>
<tr>
<td>( \sigma_A \geq 0 \geq \sigma_B ) and ( \frac{\sigma_B}{\sigma_A} \leq 1 )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} )</td>
</tr>
<tr>
<td>( \sigma_A \geq 0 \geq \sigma_B ) and ( \frac{\sigma_B}{\sigma_A} &gt; 1 )</td>
<td>( \frac{(S_{uc} - S_{ut})}{S_{uc} S_{ut}} \sigma_A - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} )</td>
</tr>
<tr>
<td>( 0 \geq \sigma_A \geq \sigma_B )</td>
<td>( \sigma_B = -\frac{S_{uc}}{n} )</td>
</tr>
</tbody>
</table>
3- Modified II-Mohr

- This theory used for the data that are still outside the extended region that are defined by M I-M theory. M II-M using a parabolic relation:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Equation to be used</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $\sigma_A \geq \sigma_B \geq 0$</td>
<td>$\sigma_A = \frac{S_{ut}}{n}$</td>
</tr>
<tr>
<td>2 $\sigma_A \geq 0 \geq \sigma_B$ and $</td>
<td>\frac{\sigma_B}{\sigma_A}</td>
</tr>
<tr>
<td>3 $\sigma_A \geq 0 \geq \sigma_B$ and $</td>
<td>\frac{\sigma_B}{\sigma_A}</td>
</tr>
<tr>
<td>4 $0 \geq \sigma_A \geq \sigma_B$</td>
<td>$\sigma_B = \frac{S_{uc}}{n}$</td>
</tr>
</tbody>
</table>
Example 1.1

A hot-rolled steel has a yield strength of $S_y = S_{yc} = 680$ MPa and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

a) 475, 475, 0 MPa.
b) 205, 475, 0 MPa.
c) 0, 475, -205 MPa.
d) 0, -205, -475 MPa.
e) 205, 205, 205 MPa.

Solution:

Since $\varepsilon_f = 0.55$ and $S_y = S_{yc}$, the material is ductile. Hence, we can use both of maximum shear stress (MSS) and distortion energy (DE) theories.

a) $\sigma_A = \sigma_1 = 475 \text{ MPa}, \sigma_B = \sigma_2 = 475 \text{ MPa}, \sigma_3 = 0 \text{ MPa}$

MSS: $\frac{S_y}{\sigma_A} = \frac{680}{475} = 1.432$

DE: $\sigma' = \left(475^2 - 475(475) + 475^2\right)^{0.5} = 475$ MPa

$n = \frac{S_y}{\sigma'} = \frac{680}{475} = 1.432$

b) $\sigma_A = \sigma_1 = 475 \text{ MPa}, \sigma_B = \sigma_2 = 205 \text{ MPa}, \sigma_3 = 0 \text{ MPa}$

MSS: $\frac{S_y}{\sigma_A} = \frac{680}{475} = 1.432$

DE: $\sigma' = \left(475^2 - 475(205) + 205^2\right)^{0.5} = 412.64$ MPa

$n = \frac{S_y}{\sigma'} = \frac{680}{412.64} = 1.65$

c) $\sigma_A = \sigma_1 = 475 \text{ MPa}, \sigma_2 = 0 \text{ MPa}, \sigma_B = \sigma_3 = -205 \text{ MPa}$

MSS: $\frac{S_y}{\sigma_A - \sigma_B} = \frac{680}{475 - (-205)} = 1$

DE: $\sigma' = \left(475^2 - 475(-205) + (-205)^2\right)^{0.5} = 604.2$ MPa

$n = \frac{S_y}{\sigma'} = \frac{680}{604.2} = 1.13$
d) \( \sigma_1 = 0 \), \( \sigma_A = \sigma_2 = -205 \), \( \sigma_B = \sigma_3 = -475 \) MPa

MSS: \[ n = \frac{S_y}{\sigma_B} = \frac{680}{-475} = 1.432 \]

DE: \[ \sigma^* = \left( (-475)^2 - (-475)(-205) + (-205)^2 \right)^{0.5} = 412.64 \text{ MPa} \]
\[ n = \frac{S_y}{\sigma^*} = \frac{680}{412.64} = 1.65 \]

e) \( \sigma_1 = 205 \), \( \sigma_2 = 205 \), \( \sigma_3 = 205 \) MPa

MSS: \[ n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{680}{0} = \infty \]

DE: \[ \sigma^* = \left( \frac{205 - 205}{2} + \frac{(205 - 205)^2}{2} \right)^{0.5} = 0 \]
\[ n = \frac{S_y}{\sigma^*} = \frac{680}{0} = \infty \]

A summery table of the factors of safety is included for comparisons.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSS</td>
<td>1.432</td>
<td>1.432</td>
<td>1</td>
<td>1.432</td>
<td>\infty</td>
</tr>
<tr>
<td>DE</td>
<td>1.432</td>
<td>1.65</td>
<td>1.13</td>
<td>1.65</td>
<td>\infty</td>
</tr>
</tbody>
</table>
Example 1.2
The cantilevered tube shown in Figure is to be made of 2014 aluminum alloy treated to obtain specified minimum yield strength of 276 MPa. We wish to select a stock-size tube from Table A-8 using a design factor of \( n_d = 4 \). The bending load is \( F = 1.75 \) kN, the axial tension is \( P = 9.0 \) kN, and the torsion is \( T = 72 \) N.m. What is the realized factor of safety?

Solution:

The maximum stresses will be expected to be at the fixed support of cantilevered tube: Hence, the internal loading at the fixed support can be shown in figure below:

\[ \sum F_x = 0 \quad , \quad N = P \]
\[ \sum F_y = 0 \quad , \quad V = F \]
\[ \sum M_z = 0 \quad , \quad M + Fd = 0 \quad \Rightarrow \quad M = -Fd \]
\[ \sum T_x = 0 \quad , \quad T \]

The maximum stresses are at the top surface of the tube at the origin:

\[ \sigma_x = \frac{P}{A} + \frac{M_x c}{I_{zz}} = \frac{9000}{A} + \frac{(210)(d_o / 2)}{I_{zz}} \]

The torsional stress at the same point is:

\[ \tau_{zz} = \frac{T_r}{J} = \frac{72(d_o / 2)}{J} \]
Using the Distortion-energy theory (Von Mises Theory):
\[
\sigma' = \left(\sigma_x^2 + 3\tau_{xz}^2\right)^{1/2} \quad \text{where} \quad \sigma_y = 0
\]

On the basis of the given design factor, the goal for \(\sigma'\) is:
\[
\sigma' \leq \frac{S_y}{n_d} = \frac{276}{4} = 69 \text{ MPa}
\]

Using table A-8 reveals that a 42mm X 5mm tube is satisfactory which gives the following data: \(d_o=42\text{mm}, t=5\text{mm}, A=5.809\text{cm}^2, J=10.13\text{cm}^4, J=20.255\text{cm}^4\)

Hence,
\[
\sigma_x = \frac{p}{A} + \frac{M \cdot c}{I_{zz}} = \frac{9000}{5.809 \times 10^{-4}} + \frac{(210)(0.042/2)}{10.13 \times 10^{-8}} = 59 \text{ MPa}
\]

The torsional stress at the same point is:
\[
\tau_{xz} = \frac{Tr}{J} = \frac{72(d_o/2)}{J} = \frac{72(0.021/2)}{20.255 \times 10^{-8}} = 7.46 \text{ GPa}
\]

Using the Distortion-energy theory (Von Mises):
\[
\sigma' = \left(\sigma_x^2 + 3\tau_{xz}^2\right)^{1/2} = 60.4 \text{ MPa}
\]

Thus, the realized factor of safety:
\[
n = \frac{S_y}{\sigma'} = \frac{276}{60.4} = 4.57
\]

If we select the next smaller tube size: 42x4-mm to check if it is applicable for the design or not we have the following data:
\(\sigma' = 71.05\text{MPa}, \) which is above the design factor \(\sigma' \leq 69\text{MPa}\) and the factor of safety for this value:
\[
n = \frac{S_y}{\sigma'} = \frac{276}{71.05} = 3.88
\]