Chapter 8: Frequency Domain Analysis

By Dr. Ridha Jemal

Electrical Engineering Department
College of Engineering
King Saud University
1431-1432

8.1. Introduction to Nyquist Criterion
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The Routh-Hurwitz method discussed in previous chapter, is useful for investigating the characteristic equation expressed in terms of the complex variable \( s = \sigma + j\omega \).

We see how frequency response methods are used to investigate stability. In this chapter, we will investigate the stability of a system in the real frequency domain.

A frequency domain stability criterion was developed by H. Nyquist in 1932 and remains fundamental approach for the stability of linear system.

We introduce the following concepts: gain margin; phase margin; bandwidth. Frequency response stability result-known as the Nyquist stability criterion is presented.

The Nyquist stability criterion determines the stability of closed-loop system from its open-loop frequency response and open-loop poles.
Nyquist Stability Criterion

- Consider the system of the following figure. Let us establish the four concepts:
  1. The relationship between the poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$
  2. The relationship between the zeros of $1 + G(s)H(s)$ and the poles of C.L.T.F $T(s)$.
  3. The concept of mapping points.
  4. The concept of mapping contours.

- Let $G = \frac{N_1}{D_1}$ and $H = \frac{N_2}{D_2}$

The closed-loop transfer function is:

$$G(s)H(s) = \frac{N_1N_2}{D_1D_2}$$

$$1 + G(s)H(s) = 1 + \frac{N_1N_2}{D_1D_2} = \frac{D_1D_2 + N_1N_2}{D_1D_2}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_1D_2}{N_1N_2 + D_1D_2}$$

- We conclude that the poles of $1 + G(s)H(s)$ are the same as the poles of $G(s)H(s)$, the open-loop system.
- The zeros of $1 + G(s)H(s)$ are the same as the poles of $T(s)$, the closed-loop system.
Nyquist Stability Criterion

- The term mapping is defined as the substitution of a complex number into a function, F(s), another complex number results.
- The contour mapping concept is a collection of points to form a closed curve.
- The Nyquist stability criterion is based on two concepts from complex variable theory, contour mapping and the Principle of the Argument.

Contour Mapping

- **Contour A** can be mapped through F(s) into **contour B** by substituting each point of contour A into the function F(s) and plotting the resulting complex numbers.

For example, point X in the following figure maps into X’ through the function F(s).
Nyquist Stability Criterion

Contour Mapping

- If we assume a clockwise direction for mapping the points on contour A, the contour B maps a clockwise direction if $F(s)$ has just zeros or has just poles that are not encircled by the contour.

\[ F(s) = (s - z_1) \]

\[ F(s) = \frac{1}{(s - p_1)} \]
Nyquist Stability Criterion

Contour Mapping

- The contour $B$ maps in a counterclockwise direction if $F(s)$ has just poles that are encircled by the contour. Also, you should verify that, if the pole or zero of $F(s)$ is enclosed by contour $A$, the mapping encircles the origin.

\[
F(s) = \frac{1}{s-p_1}
\]

\[
F(s) = \frac{s-z_1}{s-p_1}
\]
Nyquist Stability Criterion

Contour Mapping: Application to the Nyquist Criterion

- Let us first assume that \( F(s) = 1 + G(s)H(s) \) with the picture of poles and zeros of \( 1 + G(s)H(s) \) near a contour \( A \).

- As each point \( Q \) of the contour \( A \) is substituted into \( 1 + G(s)H(s) \), a mapped point results on contour \( B \). Assuming that \( F(s) \) has two zeros and three poles.
To investigate the stability of a control system, we consider the characteristic equation, which is $\Delta(s) = 0$. For a system to be stable, all the zeros of $\Delta(s)$ must lie in the left-hand $s$-plane.

**Argument Principle**

$Z =$ Number of zeros of $1+G(s)H(s)$ in the right-half $s$-plane  
$N =$ Number of clockwise encirclements of the $-1+j0$  
$P =$ Number of poles of $1+G(s)H(s)$ in the right-half $s$-plane  
$\Delta(s) = 1+G(s)H(s)= 1+ L(s)$

$$\Delta(s) = 1 + L(s) = \prod_{i=1}^{n} (s - s_i) = 0$$

$$\prod_{k=1}^{m} (s + s_k)$$

$N = Z - P$

The closed-loop system is stable if and only if $Z = 0$. 

**Nyquist Stability Criterion**

- The term mapping is defined as the substitution of a complex number into a function, $F(s)$, another complex number results.
- The contour mapping concept is a collection of points to form a closed curve.
- The Nyquist stability criterion is based on two concepts from complex variable theory, contour mapping and the Principle of the Argument.
Nyquist Stability Criterion

The Nyquist criterion is a direct application of the principle of argument when the Nyquist path is used. Since $\Delta(s) = 1+L(s)$, its origin is the point $(-1,0)$ which is called the critical point.

We choose a contour $\Gamma_s$ in the $s$-plane that encloses the entire right-hand $s$-plane, and we determine whether any zeros of $\Delta(s)$ lie within $\Gamma_s$.

A closed-loop system is stable if and only if the contour $\Gamma_L$ in the $s$-plane does not encircle the $(-1,0)$ point when the number of poles of $L(s)$ in the right-hand $s$-plane is zero ($P=0$).

A closed-loop system is stable if and only if the contour $\Gamma_L$, the number of counterclockwise encirclements of the $(-1,0)$ point is equal to the number of poles of $L(s)$ with positive real parts ($Z=0 \Rightarrow N=-P$).
To determine $N$ with respect to the origin (or any point), we have to **draw a line** from the point in any direction to a point **as far as necessary**. The number of NET intersections of this line with the locus gives the magnitude of $N$. 

\[ N = -2 \]
Some important properties of the Nyquist stability criterion are:

1. It provides a necessary and sufficient condition for closed-loop stability based on the open-loop transfer function.

2. The reason the -1 point is so important can be deduced from the characteristic equation, \( \Delta(s) = 1 + L(s) = 0 \). This equation can also be written as \( L(s) = -1 \).

3. Most process control problems are open-loop stable. For these situations, \( P = 0 \) and thus \( Z = N \). Consequently, the closed-loop system is unstable if the Nyquist plot for \( L(s) \) encircles the -1 point, one or more times.

4. A negative value of \( N \) indicates that the -1 point is encircled in the opposite direction (counter-clockwise). This situation implies that each countercurrent encirclement can stabilize one unstable pole of the open-loop system.
Nyquist Stability Criterion

Minimum-Phase System and Non minimum Phase System

Transfer function having neither poles nor zeros in the right-half plane are minimum-phase transfer function

System having poles and/or zeros in the right-half s-plane is non minimum-phase system

Closed-loop Stability of minimum-phase System

For minimum phase system, \(N = 0\). Therefore, we need only to see whether the \((-1, j0)\) is enclosed or not. We do not care how many times it is enclosed

Example 1:
Let us consider:
\[
L(s) = \frac{K}{s(s + 3)(s + 7)}
\]
Substitute \(s = j\omega\)
\[
L(j\omega) = \frac{K}{j\omega(j\omega + 3)(j\omega + 7)}
\]
\[
L(j0) = \infty \angle -90^\circ - \tan^{-1} \left( \frac{\omega}{3} \right) - \tan^{-1} \left( \frac{\omega}{7} \right) = \infty \angle -90^\circ
\]
\[
L(j\infty) = 0 \angle -90^\circ - 90^\circ - 90^\circ = 0 \angle -270^\circ
\]
Nyquist Stability Criterion

The intersection with the real axis if any is given by:

\[ L(j\omega) = \frac{K}{j\omega(-\omega^2 + 10j\omega + 21)} = \frac{K}{-10\omega^2 + j(21\omega - \omega^3)} \]

\[ L(j\omega) = \frac{K[-10\omega^2 - j(21\omega - \omega^3)]}{[-10\omega^2 + j(21\omega - \omega^3)][-10\omega^2 - j(21\omega - \omega^3)]} = \frac{K[-10\omega^2 - j(21\omega - \omega^3)]}{100\omega^4 + (21\omega - \omega^3)^2} \]

\( L(j\omega) \) intersects with the real axis (\( \text{Im}(L(j\omega)) = 0 \)):

\[ \frac{K\omega(21-\omega^2)}{100\omega^4 + (21\omega - \omega^3)^2} = \frac{K(21-\omega^2)}{\omega[100\omega^2 + (21-\omega^2)^2]} \Rightarrow \omega = \infty \text{ or } \omega = \sqrt{21} \text{ (we need positive } \omega) \]

\[ \text{Re}[L(j\omega)] = \frac{-10K}{100\omega^2 + (21-\omega^2)^2}, \text{Re}[L(j\sqrt{21})] = -0.00476K \]

\[ L(j\sqrt{21}) = -1 = -0.00476K \Rightarrow K = 210 \]
The simplified Nyquist Criterion can be applied for both Minimum and non minimum phase systems:

Consider $\Delta(s) = 1 + G(s)H(s) = 1 + L(s)$ the C.E. of the closed-loop control system

$Z =$ Number of zeros of $1 + G(s)H(s)$ in the right-half $s$-plane
$P =$ Number of poles of $1 + G(s)H(s)$ in the right-half $s$-plane
$P_0 =$ Number of poles of $L(s)$ or $1 + L(s)$ on the $j\omega$ axis
$\phi_{11} =$ Angle traversed by the Nyquist plot with respect to (-1,j0) point

$$\phi_{11} = \left[Z - P - 0.5P_0\right].180^\circ \quad (1)$$

For closed-loop stability: $Z = 0$

$$\phi_{11} = -\left[P + 0.5P_0\right].180^\circ \quad (2)$$
Simplified Nyquist Criterion

Remarks:

1. With this method, we need to plot the Nyquist plot only for \( s=j\infty \) to \( s=0 \)

2. \( \phi_{11} \) is the angle variation which is a directed angle (\( \phi_{11} =-270^\circ \) is not the same as \( \phi_{11} =90^\circ \))

3. If \( \phi_{11} \) is positive, it correspond to the \((-1,j0)\) being enclosed

4. Since \( P \) and \( P_\omega \) are positive numbers, Equation 2 indicates that if \( \phi_{11} \) is positive, the closed-loop system is unstable.

5. For system with minimum-phase loop transfer functions, \( P=0 \). Equation 2 becomes:

\[
\phi_{11}=-P_\omega\cdot90^\circ
\]

6. For system with non-minimum-phase loop transfer functions, Even if the \((-1,j0)\) is not enclosed, the equation 1 must be satisfied for the closed-loop system to be stable.