Chapter 7 : Root Locus Technique

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7.1. Introduction
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In previous chapters, we have demonstrated the important of poles and zeros of the closed loop transfer function of a linear control system on the dynamic performance of the system.

- The behavior of the roots of the CE (poles of the c.l.t.f) will reveal the relative and the absolute stability
- The transient behavior of the system is governed by the zeros of the closed loop-transfer function

The root locus is a plot of the trajectories of the roots of the characteristic equation when a certain parameter varies

- The root locus is the locus of the roots of the characteristic equation of the closed-loop transfer function as the loop gain of the feedback system is increased from zero to infinity. Traditionally, the root locus is constructed with the loop gain $K$ as the variable.
- The locus of roots indicates the contribution of each open-loop pole and zero to the locations of closed-loop poles.
Basics on the Loci

Consider the following control systems:

1. **Let** \( G = \frac{N}{D} \)

   The closed-loop transfer function is:

   \[
   Y(s) = \frac{KG}{1 + KG}
   \]

2. **Let** \( G = \frac{N1}{D1} \) and \( H = \frac{N2}{D2} \)

   The closed-loop transfer function is:

   \[
   Y(s) = \frac{KG}{1 + KGH} = \frac{KN1 / D1}{1 + (KN1N2 / D1D2)}
   \]

\[ \frac{Y}{R} = \frac{KN1D2}{D1D2 + KN1N2} \]
Note that the two cases above that:

- The zeros of any closed-loop system are the zeros of the open loop.
- The poles are not as evident, the depend on $K$, $G$ and $H$.

Since there are many specifications that the control system should satisfy (stability, %-Overshoot, delay time, rise time, settling time and other specification that we did not cover), we should look for the value of $K$ satisfies these specifications.

- Routh-Hurwitz determine the Range of $K$ for which the system is stable.
- Root Locus Technique determine $K$ to meet the rest of the specification.

The characteristic equation is expressed as: $1 + KGH = 0$

For the Unity Feed back ($H=1$) $\Rightarrow$ $KG = -1$

$KG(s)=1 \angle \pm 180 \pm 360^\circ$ (magnitude = 1; angle =180°)
Basics on the Loci

**Example 1**

\[ G(s) = \frac{1}{s+2} \quad \text{and} \quad KG(s) = \frac{K}{s+2} \]

Let \( s = \alpha + j\omega \Rightarrow s+2 = (\alpha+2) + j\omega \)

\[ KG(s) = \frac{K}{\sqrt{(\alpha+2)^2 + \omega^2}} \angle \tan^{-1} \frac{\omega}{\alpha+2} \]

\[
\begin{aligned}
\tan^{-1} \frac{\omega}{\alpha+2} &= 180^\circ \quad (1) \\
\frac{K}{\sqrt{(\alpha+2)^2 + \omega^2}} &= +1 \quad (2)
\end{aligned}
\]

\[
\begin{aligned}
\frac{\omega}{\alpha+2} &= \tan(180^\circ) = 0 \Rightarrow \omega = 0, \text{the pole is in the real axis} \\
\frac{K}{\alpha+2} &= -1 \Rightarrow K = -(\alpha+2) \Rightarrow \alpha = -K - 2
\end{aligned}
\]

<table>
<thead>
<tr>
<th>( K )</th>
<th>( -10 )</th>
<th>( -6 )</th>
<th>( -2 )</th>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>( K=6 )</td>
<td>( K=4 )</td>
<td>( K=0 )</td>
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Basics on the Loci

Example 2
Consider \( G(s) = \frac{s + 3}{s - 1} \). There is a pole at \( s=1 \).

The system is unstable. Suppose we want to make it stable and we want to find the poles of the closed-loop as a function of \( K \) using the unity feedback.

\[
\frac{Y}{R} = \frac{KG}{1 + KG} = \frac{K(s + 3)}{(s - 1) + K(s + 3)} = \frac{K(s + 3)}{(K + 1)s + (3K - 1)}
\]

One zero at: \( s=-3 \)
One pole at: \( s = 1 - 3K/K+1 \)

\[
\begin{array}{c|cccc}
\text{End} & K=\infty & K=3 & K=1 & K=1/3 & K=0 \\
\hline
-3 & x & x & x & x & x \\
\end{array}
\]

The poles start when \( K=0 \) at the open-loop pole (+1) and end when \( K=\infty \) at the open-loop zero (\( s=-3 \)). Let’s apply the Routh-Hurwitz:

\[
\begin{array}{c|cc}
S^1 & K+1 & 0 \\
S^0 & 3K-1 & 0 \\
\end{array}
\]

For stability \( 3K - 1 > 0 \iff K > 1/3 \)

The Routh-Hurwitz Criterion indicates the range of \( K \) for stability and does not determine the exact value of \( K \). The root locus shows the pole location (Starts at the open-loop pole and terminates at the open-loop zero, including zero at \( \infty \) if the system has zero at infinity).
Characteristics of the Loci

The following properties of the root Loci are introduced for the purpose of constructing the root Loci. These properties are developed based on the relation between the poles and zeros of $G(s)H(s)$ and the zeros of $1+G(s)H(s)$

1. The Loci starts $(K=0)$ at the poles of $G(s)$ and terminates $(K=\infty)$ at the zeros of $G(s)$ at infinity

   **Proof**: Let $G(s)=\frac{KN(s)}{D(s)}$, the C.E. of the c.l. system is $1+G(S) = 0 \Rightarrow 1+ \frac{KN(s)}{D(s)} = 0 \Rightarrow \left| \frac{N(s)}{D(S)} \right| = \frac{1}{|k|}$

   i. $K=0$ $1/K=\infty$ $\Rightarrow$ The solution is the poles of $G(s)$ $\Rightarrow$ $D(s)=0$

   ii. $K=\infty$ $1/K=0$ $\Rightarrow$ $N(s) = 0$ $\Rightarrow$ The solution is the zeros of $G(S)$ including the ones at infinity

2. The Loci exits on the real axis only to the left of an odd number of real poles and zeros of $G(s)$

3. If the loci leaves the real axis, it must be symmetrical about this axis because complex poles appear in complex conjugate pairs
Characteristics of the Loci

4. Loci which terminates at \( \infty \) must follow asymptotes which make an angle \( \theta \) with the real axis where:

\[
\theta = \pm \frac{180^\circ (2k + 1)}{Z - P}
\]

\( Z \): number of finite zeros of \( G(s) \);
\( P \): number of poles; \( k=0,1,2, \ldots \)

5. The asymptotes intersect at point \( \sigma \) on the real axis given by:

\[
\sigma = \frac{\sum P - \sum Z}{P - Z}
\]

\( \sum P \): Sum of the poles of \( G(s) \)
\( \sum Z \): Sum of the zeros of \( G(s) \)
\( P, Z \): The number of poles and zeros of \( G(s) \)

6. The point of Breakaway from the real axis can be determined by solving the equation \( \frac{dK}{ds} = 0 \) (where \( 1 + G(s) = 0 \) must be satisfied).

7. The point at which the loci cross the imaginary axis can be determined by letting \( s = j\omega \) in the characteristic equation.
Characteristics of the Loci

Example

Consider: \( G(s) = \frac{K}{s(s + 4)(s + 10)} \)

1. There are no finite zeros. Therefore, all loci must terminate at infinity.

2. The loci must exit on the real axis in the regions:
   \[ -4 < s < 0 \quad \text{and} \quad s < -10 \]

3. The loci must be symmetrical when the have to leave the real axis.

4. The angle of the asymptotes must be:
   \[ \theta = \pm \frac{180^\circ (2k + 1)}{0 - 3} = \pm 60^\circ (2k + 1) = \pm 60^\circ \quad \text{and} \quad \pm 180^\circ \]

5. The asymptote will intersect the real axis at:
   \[ \sigma = \frac{\sum P - \sum Z}{P - Z} = \frac{0 - 4 - 10}{3 - 0} = -14/3 \]

5. The point of breakaway from the real axis can be determined from:
   \( \frac{dK}{ds} = 0 \)
Characteristics of the Loci

Example

6. The C.E. is given by $1+G(s)=0$
   $K+s(s+4)(s+10)=0$
   
   \[
   \frac{dK}{ds} = 3s^2 + 28s + 40 = 0
   \]
   
   $s = -7.57$ (not valid because it does not belong to the loci)
   or $s = -1.76$

7. The point at which the loci cross the imaginary axis is given
   by replacing $s=j\omega$ in the C.E.

   $\omega = 0$ and $\omega = \sqrt{40}$
   $K = 0$  \hspace{1cm} K = 560