Question One (Dr. asmaa+dr.azza)

a) Prove the following:

i) Every Euclidean domain is a principal ideal domain.

ii) Let $R$ be Euclidean domain with valuation $\delta$, an element $0 \neq a \in R$ is a unit if and only if $\delta(a) = \delta(1_R)$.

iii) Every finite integral domain is a field.

iv) Let $R$ be a commutative ring with an identity. Show that $1 + ax$ is an invertible in $R[x]$ if and only if $\exists n > 0$ with $a^n = 0$

b) Find the following:

i) The factorization of the polynomial $x^4 + 3x^3 + 2x + 4$ in $\mathbb{Z}_5[x]$.

ii) $q(x)$ and $r(x)$ as described by the division algorithm so that $f(x) = g(x)q(x) + r(x)$ with $\deg(r(x)) < \deg(g(x))$ where $f(x) = x^5 - 2x^4 + 3x - 5$ and $g(x) = 2x + 1$ in $\mathbb{Z}_{11}[x]$.

Question Two

a) i) Define maximal ideal in a ring.

ii) Let $R$ be a principal ideal domain. Prove that a nontrivial ideal $<a>$ of $R$ is a prime ideal in $R$ if and only if it is a maximal ideal in $R$.

b) Let $F$ be a field of quotients of $D$ and let $L$ be any field containing $D$. Then prove that there exists a map $\psi : F \rightarrow L$ that gives an isomorphism of $F$ with a subfield of $L$ such that $\psi(a) = a$ for $a \in D$.

[Hint define $\psi$ by $\psi(a b) = \psi(a) \psi(b)$].

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Question Three

a) (i) Define an irreducible element in a ring.

(ii) Let $R$ be a principal ideal domain. Prove that a nonzero element $p \in R$ is irreducible if and only if it is a prime element in $R$.

b) Let $a, b, c$ be the elements of the principal ideal ring $R$. If $c \mid ab$ with $a$ and $c$ relatively prime, then prove that $c \mid b$.

c) Make true or false with reasons

(i) If $f$ be a homomorphism from the field $F$ into the field $F'$, then either $f$ is the trivial homomorphism or else $f$ is one-to-one.

(ii) Any two irreducibles in any unique factorization domain are associates.

(iii) Every quotient ring of an integral domain is an integral domain.

(iv) $2x + 4$ is irreducible in $\mathbb{Q}[x]$ but not in $\mathbb{Z}[x]$.

(v) Every nonzero element of an integral domain $D$ is a unit in a field $F$ of quotient of $D$.

Question Four

a) i) Define primitive polynomial in $R[x]$, the ring of all polynomials over a ring $R$.

ii) Let $R$ be a unique factorization domain. If $f(x), g(x)$ are both primitive polynomials in $R[x]$, then prove that their product $f(x)g(x)$ is also primitive in $R[x]$.

b) Let $R$ be an integral domain and $f(x) \in R[x]$ be a nonzero polynomial of degree $n$. Prove that $f(x)$ can have at most $n$ distinct roots in $R$.

⇒ With best wishes

Dr. Asma and Dr. Azza
Question One

a) Find the following:
i) The factorization of the polynomial \( x^4 + 3x^3 + 2x + 4 \) in \( \mathbb{Z}_5[x] \).
ii) \( q(x) \) and \( r(x) \) as described by the division algorithm so that
\( f(x) = g(x) q(x) + r(x) \) with \( \text{deg}(r(x)) < \text{deg}(g(x)) \) where
\( f(x) = x^5 - 2x^4 + 3x - 5 \) and \( g(x) = 2x + 1 \) in \( \mathbb{Z}_{11}[x] \).
iii) A commutative ring \( R \) with an identity for which \( R[x] \) has zero divisors.

b. Determine whether the polynomial \( 8x^3 + 6x^2 - 9x + 24 \) in \( \mathbb{Z}[x] \)
satisfies an Eisenstein test for irreducibility over \( \mathbb{Q} \).

Question Two

Prove the following:

a) If \( R \) be a unique factorization domain, with field of quotients \( K \)
and if \( f(x) \in K[x] \) be a nonconstant polynomial with \( \text{deg} f(x) > 0 \)
then there exist non zero elements \( a, b \in R \) and a primitive polynomial
\( f_1(x) \in R[x] \) such that \( f(x) = ab^{-1}f_1(x) \).
Furthermore, \( f_1(x) \) is unique up to invertible elements of \( R \) as factors.

b) If \( R \) is a principal ideal domain then every noninvertible element
\( 0 \neq a \in R \) has a unique factorization into a finite product of primes.

c) If \( F \) is a field then \( f(x) \) is an irreducible polynomial in \( F[x] \) if and
only if the quotient ring \( F[x]/<f(x)> \) forms field.
Question Three

Determine (giving a proof or a counter example) which of the following statements are true.

i) Every quotient ring of an integral domain is an integral domain.

ii) The invertible elements in an integral domain form a cyclic group under multiplication.

iii) If $R = M_{2 \times 2}(\mathbb{R})$ is not a commutative ring then the nilpotent elements do not necessarily form an ideal in $R$.

iv) If $f(x) \in R[x]$ has constant term $1$, then $f(x)$ is not a zero divisor in $R[x]$?

Question Four

Prove the following:

i) If $R$ be a commutative ring with an identity then $1+ax$ is an invertible in $R[x]$ if and only if $\exists n > 0$ with $a^n = 0$.

ii) Every Euclidean domain is a principal ideal domain.

iii) If $f(x) \in \mathbb{R}[x]$ with $\deg f(x) > 0$ then $f(x)$ can be factored into linear and irreducible quadratic factors.

iv) If $R$ is an integral domain then $0 \neq p \in R$ is an irreducible element of $R$ if and only if $\langle p \rangle$ is a maximal principal ideal.