Department of Mathematic Faculty of Science King Saud University	Final Term Exam Complex Analysis PM 385	21 June 2010 Time allowed:Three hours

Answer only four questions of the following

Question (1)

1. (3 degrees) Find all the solutions of the equation $e^{4z} = 1$.

2. (10 degrees) Let $f(z) = u(x, y) + iv(x, y)$ be defined in some open set G containing the point z_0.

   (a) [4] State the necessary and the sufficient conditions that should be satisfied to guarantee that f is differentiable at z_0.

   (b) [3] Prove that if f is an analytic function in a domain G such that Im$f(z)$ is constant, then $f(z)$ is identically constant.

   (c) [3] Prove that if $f(z)$ is analytic and real-valued in a domain G, then $f(z)$ is constant in G

Answers

1. $z = \frac{i\pi k}{2},\ k = 0, 1, 2, 3.$

2. (a) The necessary condition: If f is differentiable at z_0, then the first partial derivatives of u and v exist and satisfy the Cauchy Riemann equations at z_0 which are

$$u_x(z_0) = v_y(z_0), \quad u_y(z_0) = -v_x(z_0).$$

   The sufficient condition: If the first partial derivatives of u and v exist and continuous at z_0 satisfy the Cauchy Riemann equations, then f is differentiable at z_0.

   (b) Assume that $f(z) = u(x, y) + iC$, where C is a real number and $z$lies in G. Since the Cauchy Riemann equations hold at each point in G, then $u_x = v_y = 0$ at every point in G. Consequently u is identically equal to constant.

   (c) since f is real valued, hence $Imf = 0$, so from (b), f is identically constant.

Question (2)

4 Prove that the functions $u(x, y) = e^x \sin y$ is a harmonic function and find a harmonic conjugate $v(x, y)$.

3 Prove that the set $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$ is open subset of $\mathbb{C}$. Find its boundary set.

3 Find the Laurent series of the function $f(z) = \dfrac{z}{(z + 1)(z - 2)}$ in the following domains

$$\text{(i)}|z| < 1 \qquad \text{and} \qquad \text{(ii)}1 < |z| < 2.$$

3 Find and classify the isolated singularities of the following functions

$$\text{(i)}\ \frac{\tan z}{z} \qquad \text{(ii)}\ ze^{1/z} \qquad \text{(iii)}\ \frac{\sin z}{z^2 - 1}.$$

Answers

1. First verify that $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. To calculate the harmonic conjugate. Observe that the harmonic conjugate function should satisfy the equation
$$v_y = u_x = e^x \sin y, \text{and } v_x = -u_y = -e^x \cos y.$$
Thus after simple calculation you shall get $x = -e^x \cos y + C$, where C is an arbitrary constant.

2. We prove that each point in A is an interior point. So take $z_0 \in A$. Let
$$\delta < \min\{1 - |z_0|, |z_0|\}$$
We prove that
$$D(z_0, \delta) \subset A.$$
Let $z \in A$. Hence $|z - z_0| < \delta$, but we want to prove that $0 < |z| < 1$. Hence
$$|z| > |z_0| - |z - z_0| > |z_0| - \delta > 0,$$
and
$$|z| < |z_0| + |z - z_0| < |z_0| + \delta < 1.$$
Hence $z_0 \in A$. The boundary set, $\delta(A) := \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}$. We should give a proof for that.

3. Now
$$\frac{z}{(z+1)(z-2)} = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}. \tag{1}$$
For $|z| < 1$, we have
$$\frac{1}{z+1} = \sum_{n=0}^{\infty}(-z)^n, \quad \frac{1}{z-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$
Substituting in (1) gives
$$\frac{z}{(z+1)(z-2)} = \frac{1}{3}\sum_{n=0}^{\infty}((-1)^n - 2^{-n})z^n$$
In case of $1 < |z| < 2$, we have
$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = z^{-1}\sum_{n=0}^{\infty}(-z)^{-n}.$$
Hence
$$\frac{z}{(z+1)(z-2)} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^{-n-1} - 2^{-n})z^n$$

4. (i) The function $f(z) := \frac{\tan z}{z}$ has an isolated singularities at $z = 0$ and poles at the points $z = (n + 1/2)\pi$, $n \in \mathbb{Z}$.
   (ii) The function $ze^{1/z}$ has an essential singularity at $z = 0$.
   (iii) The function $\frac{\sin z}{z^2-1}$ has simple poles at $z = \pm 1$.

Question (3)

Compute the following integrals.

2 $\int_C ze^{\frac{1}{z}}\, dz$ counter clockwise around the circle $|z| = 2$.

2 $\int_\Gamma \frac{1}{z}\, dz$ for any contour in the **left** half plane from $z = -3i$ to $z = 3i$.

3 $\int_C \dfrac{z^3}{(z+1)^3}\, dz$, counter clockwise around the circle $|z| = 2$.

2 $\int_C \dfrac{\sinh z}{z^2+4}\, dz$, where C is the negatively oriented circle $|z| = 1$.

4 $\int_0^{2\pi} \dfrac{d\theta}{2+\sin\theta}$.

Answers:

1. $\int_C ze^{1/z}\, dz = \sum_{n=0}^{\infty} \int_C \frac{z^{-n+1}}{n!}\, dz = \frac{1}{2!}\int_C z^{-1}\, dz = i\pi$

2. $$\begin{aligned}\int_\Gamma \frac{1}{z}\, dz &= \mathcal{L}_0(3i) - \mathcal{L}_0(-3i)\\ &= \text{Log}|3| + i\frac{\pi}{2} - \left(\text{Log}|3| - \frac{3\pi}{2}\right) = -i\pi.\end{aligned}$$

3. From the generalized Cauchy integral formula, Since z^3 is analytic on the bounded domain bounded by C, then $\int_C \dfrac{z^3}{(z+1)^3}\, dz = \frac{d^2}{dz^2} z^3|_{z=-1} = -6\pi$

4. $\int_C \dfrac{\sinh z}{z^2+4}\, dz = 0$

5. Take $z = e^{i\theta}$, then $\sin\theta = (z - z^{-1})/2i$. Hence $\frac{dz}{iz} = d\theta$ and the integral, say I, is equal to
$$2\int_C \frac{dz}{z^2+4iz-1}$$
where C is positively oriented unit circle. Now complete the solution by using the residue theorem to get
$$I = 4i\pi \lim_{z\to a} \frac{(z-a)}{(z-a)(z-b)} = \frac{4i\pi}{a-b},$$
where $a = -2+\sqrt{3}i$ and $b = -2-\sqrt{3}i$. So $I = \dfrac{2i\pi}{\sqrt{3}}$.

Question (4)
Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$. Prove that

3 $a_k = \dfrac{\frac{d^k}{dz^k} P(z)|_{k=0}}{k!} = \frac{P^{(k)}(0)}{k!}$ for $k = 0, 1, \ldots, n$.

2 Use the Cauchy integral formula to represent $P^{(k)}(0)$.

4 If $\max P(z) = M$ for $|z| = 1$. Show that $|a_k| \leq M$ for $k = 0, 1, \ldots, n$.

4 There is an $R > 0$ so that if C is the circle $|z| = R$ positively oriented, then

$$\int_C \frac{P'(z)}{P(z)}\, dz = 2n\pi\, i.$$

Answers:

1. Since $\frac{d^k}{dz^k} P(z) = \sum_{l=k}^{n} l(l-1)\ldots(l-k+1)a_l z^{l-k}$. Hence

$$\frac{d^k}{dz^k} P(z)|_{z=0} = a_k\, k!,$$

which proving 1.

2. $P^{(k)}(0) = \frac{k!}{2\pi\, i} \int_C \frac{P(z)}{z^{k+1}}\, dz$, where C is any positively oriented contour containing zero.
3. $|P^{(k)}(0)| \leq \frac{k!}{2\pi}$M l(C)=K!M, thus from 1, $a_k \leq M$.
4. $$\int_C \frac{P'(z)}{P(z)} = N - P,$$

where N is the number of zeros inside N and P is the number of poles inside C. Since $P = 0$, and $P(z)$ is entire, then we can choose R large enough such that the circle C contains all the zeros of the polynomial Z. So in this case $N = n$ and we get the answer of the problem.

Question (5)

1. Prove the following theorem. If $f(z)$ is analytic in the disk $|z - z_0| < R$, then the Taylor series

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j,$$

converges to $f(z)$ for all z in the disk. Furthermore, the convergence of the series is uniform in any closed subdisk $|z - z_0| \leq R' < R$.

Answer Look at Saff and Snider's book.

The end of the answer sheet
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