Department of Mathematic Faculty of Science King Saud University	Final Term Exam Complex Analysis PM 385	21 June 2010 Time allowed:Three hours

Answer only four questions of the following

Question (1)

1. Find all the solutions of the equation $e^{4z} = 1$.
2. Let $f(z) = u(x, y) + iv(x, y)$ be defined in some open set G containing the point z_0.
   (a) State the necessary and the sufficient conditions that should be satisfied to guarantee that f is differentiable at z_0.
   (b) Prove that if f is an analytic function in a domain G such that $\text{Im}f(z)$ is constant, then $f(z)$ is identically constant.
   (c) Prove that if $f(z)$ is analytic and real-valued in a domain G, then $f(z)$ is constant in G

Question (2)

1. Prove that the functions $u(x, y) = e^x \sin y$ is a harmonic function and find a harmonic conjugate $v(x, y)$.
2. Prove that the set $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$ is open subset of $\mathbb{C}$. Find its boundary set.
3. Find the Laurent series of the function $f(z) = \dfrac{z}{(z+1)(z-2)}$ in the following domains
$$\text{(i)}|z| < 1 \qquad \text{and} \qquad \text{(ii)}1 < |z| < 2.$$
4. Find and classify the isolated singularities of the following functions
$$\text{(i)} \frac{\tan z}{z} \qquad \text{(ii)} ze^{1/z} \qquad \text{(iii)} \frac{\sin z}{z^2 - 1}.$$

Question (3)

Compute the following integrals.

1. $\int_C ze^{\frac{1}{z}}\, dz$ counter clockwise around the circle $|z| = 2$.
2. $\int_\Gamma \frac{1}{z}\, dz$ for any contour in the **left** half plane from $z = -3i$ to $z = 3i$.
3. $\int_C \dfrac{z^3}{(z+1)^3}\, dz$, counter clockwise around the circle $|z| = 2$.
4. $\int_C \dfrac{\sinh z}{z^2 + 4}\, dz$, where C is the negatively oriented circle $|z| = 1$.
5. $\int_0^{2\pi} \dfrac{d\theta}{2 + \sin\theta}$.

Please Look at the back of the paper exam to questions 4 and 5 $\rightrightarrows$.

Question (4)
Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$. Prove that

1. $a_k = \dfrac{\frac{d^k}{dz^k} P(z)|_{k=0}}{k!} = \frac{P^{(k)}(0)}{k!}$ for $k = 0, 1, \ldots, n$.
2. Use the Cauchy integral formula to represent $P^{(k)}(0)$.
3. If $\max P(z) = M$ for $|z| = 1$. Show that $|a_k| \leq M$ for $k = 0, 1, \ldots, n$.
4. There is an $R > 0$ so that if C is the circle $|z| = R$ positively oriented, then

$$\int_C \frac{P'(z)}{P(z)}\, dz = 2n\pi\, i.$$

Question (5)

1. Prove the following theorem. If $f(z)$ is analytic in the disk $|z - z_0| < R$, then the Taylor series

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j,$$

converges to $f(z)$ for all z in the disk. Furthermore, the convergence of the series is uniform in any closed subdisk $|z - z_0| \leq R' < R$.

The end of the questions
Good Luck
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