

On the Normality of Operators

SALAH MECHERI

King Saud University, Saudi Arabia

ABSTRACT. In this paper we will investigate the normality in (WN) and (\mathcal{Y}) classes.

Keywords and phrases. Normal operators, Operators satisfying $(Re T)^2 - |T|^2 \geq 0$, (\mathcal{Y}) classes.

2000 Mathematics Subject Classification. Primary: 47A15. Secondary: 47B20, 47A63.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H . For A in $\mathcal{B}(H)$ the adjoint of A is denoted by A^* . For any operator A in $\mathcal{B}(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions: A is hyponormal if $|A^*|^2 \leq |A|^2$ (i.e., if $[A^*, A]$ is nonnegative or, equivalently, if $\|A^*x\| \leq \|Ax\|$ for every x in H), normal if $A^*A = AA^*$, quasinormal if A^*A commutes with A , and m -hyponormal if there exists a positive number m , such that

$$m^2(A - \lambda I)^*(A - \lambda I) - (A - \lambda I)(A - \lambda I)^* \geq 0, \text{ for all } \lambda \in \mathbb{C}.$$

Let (N) , (QN) , (H) , and $(m - H)$ denote the classes constituting of normal, quasinormal, hyponormal, and m -hyponormal operators. Then

$$(N) \subset (QN) \subset (H) \subset (m - H).$$

An operator T in $\mathcal{B}(H)$ is said to be hermitian if $T = T^*$. It is well known that hermitian operators can be characterized in the following way: an operator T in $\mathcal{B}(H)$ is hermitian if and only if $\langle Tx, x \rangle$ is real. In [3] the authors gave an other characterization involving inequalities. We denote by (WN) the class of operators in $\mathcal{B}(H)$ satisfying the following inequality

$$(Re T)^2 \leq |T|^2,$$

where $ReT = \frac{(T+T^*)}{2}$ is the real part of T and we will write $ImT = \frac{(T-T^*)}{2}$ for the imaginary part of T . This class has been introduced by Fong and Istratescu [3] who conjectured:

If $T \in (WN)$ and $\sigma(T)$ (the spectrum of T) is real, then T is hermitian. It is known that if in addition T is hyponormal or $T + T^*$ commutes with TT^* or if H is finite-dimensional the conjecture holds [3, Corollary 2.2]. Notice that T is hermitian if and only if

$$(ReT)^2 \geq |T|^2. \quad (1.1)$$

By reversing the inequality (1.1) we obtain the class (WN) . This class contains the class of hyponormal operators. Indeed, T hyponormal implies that

$$(ReT)^2 + (ImT)^2 \leq |T|^2.$$

Since $(ImT)^2$ is a positive operator,

$$(ReT)^2 \leq |T|^2,$$

that is, $T \in (WN)$. In this note we will investigate the normality in (WN) .

A is said to be of class \mathcal{Y}_α for $\alpha \geq 1$ if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda)^*(A - \lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class \mathcal{Y}_1 operator A is M -hyponormal, i.e., there exists a positive number M such that

$$(A - \lambda I)(A - \lambda I)^* \leq M^2 (A - \lambda I)^*(A - \lambda I) \quad \text{for all } \lambda \in \mathbb{C},$$

and M -hyponormal operators are class \mathcal{Y}_2 (see[13]).

In [15] Weber shows that every compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$ is quasinilpotent, where $\{A\}'$ denotes the commutant of A , δ_A stands for the derivation operator generated by A , i.e., $\delta_A(X) = AX - XA$, $R(\delta_A)$ is the range (image) of δ_A and $\overline{R(\delta_A)}^w$ is its weak closure. In [7] we showed that Weber's result is a consequence of a more general result. A reasonable conjecture is the following:

(S) Let T be a compact operator. If $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then T is quasinilpotent, where $\{A^*\}'$ is the commutant of the adjoint of A .

In [7], [8] it is proved that if A is normal, subnormal, dominant or m -hyponormal, then the conjecture holds. In this paper we shall see that the conjecture holds for an interesting class of operators which includes hyponormal operators. It is clear that the class (\mathcal{Y}) contains the class of normal operators. In the following we will denote the spectrum, the point spectrum, the approximate reduced spectrum, and the approximate spectrum of an operator $A \in \mathcal{B}(H)$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ar}(A)$, and $\sigma_a(A)$ respectively (see [7]). In this paper we will investigate the normality in (\mathcal{Y}) .

2. Preliminaries

Let us list some spectral properties of operators in (WN) .

Proposition 2.1. [3] *Suppose that $T \in B(H)$ is in (WN) . Then we have*

- (i) *If λ is a real number, then $T - \lambda$ is also in (WN) .*
- (ii) *$\|(T - \lambda)^*x\| \leq 3\|(T - \lambda)x\|$, for all $x \in H$ and all real number λ .*
- (iii) *If M is an invariant subspace of T , then $T|_M$ is also in (WN) ; if furthermore, $T|_M$ is hermitian, then M reduces T .*
- (iv) *If λ is real eigenvalue of T , then the eigenspace $\ker(T - \lambda)$ reduces T .*
- (v) *If λ is real and $(T - \lambda)^n x = 0$ for some $n \geq 1$, then $(T - \lambda)x = 0$.*

Remark 2.1. *As a consequence of the previous proposition, if $(\lambda_n)_{n \in \mathbb{N}}$ is a countable sequence of real eigenvalues of T in (WN) , then $H = M \oplus M^\perp$ and $T|_M$ is hermitian where $M = \bigoplus_{n \geq 0} \ker(T - \lambda_n I)$.*

3. Normality in (\mathcal{Y}) -classes

Let us begin by the following Berberian techniques [2]: Let H be a complex Hilbert space. Then there exists an Hilbert space $H^\circ \supset H$, and an isometric $*$ -homomorphism

$$\varphi : \mathcal{B}(H) \mapsto \mathcal{B}(H^\circ) \quad (A \mapsto A^\circ)$$

preserving order, i.e., for all $A, B \in \mathcal{B}(H)$ and for all $\alpha, \beta \in \mathbb{C}$ we have:

- (1) $\varphi(A^*) = \varphi(A)^*$;
- (2) $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$;
- (3) $\varphi(I_{\mathcal{H}}) = I_{\mathcal{H}^\circ}$;
- (4) $\varphi(AB) = \varphi(A)\varphi(B)$;
- (5) $\|\varphi(A)\| = \|A\|$;
- (6) $\varphi(A) \leq \varphi(B)$ if $A \leq B$;
- (7) $\sigma(\varphi(A)) = \sigma(A)$, $\sigma_a(A) = \sigma_a(\varphi(A)) = \sigma_p(\varphi(A))$;
- (8) if A is a positive operator, then $\varphi(A^\alpha) = |\varphi(A)|^\alpha$ for all $\alpha > 0$.

Lemma 3.1. *If $S \in (\mathcal{Y})$, then $\varphi(S) \in (\mathcal{Y})$.*

Proof. If $S \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2(T - \lambda I)^*(T - \lambda I), \text{ for all } \lambda \in \mathbb{C}.$$

It follows from the properties of the map φ that

$$\varphi(|TT^* - T^*T|^\alpha) \leq \varphi(k_\alpha^2(T - \lambda I)^*(T - \lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

By the condition (8) above we have

$$\varphi(|TT^* - T^*T|^\alpha) = |\varphi(TT^* - T^*T)|^\alpha,$$

for all $\alpha > 0$. Therefore

$$|\varphi(T)\varphi(T^*) - \varphi(T^*)\varphi(T)|^\alpha \leq \varphi(k_\alpha^2(T - \lambda I)^*(T - \lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

Hence $\varphi(T) \in (\mathcal{Y})$. □

Now we will present some spectral properties of the class (\mathcal{Y}) .

Theorem 3.1. *Let $S \in (\mathcal{Y})$.*

- (i) *If $\lambda \in \sigma_p(S)$, then $\bar{\lambda} \in \sigma_p(S^*)$, furthermore if $\lambda \neq \mu$, then M_λ (the proper subspace associated with λ) is orthogonal to M_μ ;*
- (ii) *If $\lambda \in \sigma_a(S)$, then $\bar{\lambda} \in \sigma_a(S^*)$;*
- (iii) *$SS^* - S^*S$ is not invertible;*
- (iv) *If M is an invariant subspace for S and $S|_M$ is normal, then M reduces S ;*
- (v) *If there exists a reducing subspace M , then $S|_M \in (\mathcal{Y})$.*

Proof. For (i) and (iv) see [13].

- (ii) Let $\mu \in \sigma_a(S)$ from the condition (7) above, we have

$$\sigma_a(S) = \sigma_a(\varphi(S)) = \sigma_p(\varphi(S)).$$

Therefore $\mu \in \sigma_p(\varphi(S))$. By applying Lemma 3.1 and the above condition (i), we get

$$\bar{\mu} \in \sigma_p(\varphi(S)^*) = \sigma_p(\varphi(S^*)).$$

Hence $\bar{\mu} \in \sigma_a(\varphi(S^*))$.

- (iii) Let $S \in (\mathcal{Y})$. Then there exists an integer $n \geq 1$ and $k_n > 0$ such that

$$\| |SS^* - S^*S|^{2^{n-1}} x \| \leq k_n^2 \| (S - \lambda I)x \|$$

for all $x \in H$, and for all $\lambda \in \mathbb{C}$.

It is known that $\sigma_a(S) \neq \emptyset$. If $\lambda \in \sigma_a(S)$, then there exists a normed sequence (x_n) in \mathcal{H} such that $\| (S - \lambda I)x_n \| \rightarrow 0$. Then $(SS^* - S^*S)x_n \rightarrow 0$ and so, $(SS^* - S^*S)$ is not invertible.

✓

Let \mathcal{A} denote a complex Banach Algebra with identity e . A state on \mathcal{A} is a functional $f \in \mathcal{A}^*$ such that $f(e) = 1 = \|f\|$. For $x \in \mathcal{A}$ let

$$W_0(x) = \{f(x) : f \text{ is a state on } \mathcal{A}\}$$

be the numerical range of x [14]. $W_0(x)$ is a compact convex set containing $\text{co}\sigma(x)$ (the convex hull of the spectrum of x) [1].

For the case $\mathcal{A} = \mathcal{B}(H)$, if $A \in \mathcal{B}(H)$ then $W_0(A) = \overline{W(A)}$, where

$$W(A) = \{(Ah, h) : h \in H, \|h\| = 1\}$$

is the special numerical range of \mathcal{A} . An element $a \in \mathcal{A}$ is finite if $0 \in W_0(ax - xa)$ for each x in \mathcal{A} ; $\mathcal{F}(A)$ (or \mathcal{F}) denotes the set of all finite elements of \mathcal{A} . It is known that \mathcal{F} contains every normal, hyponormal and dominant operator (see [7], [14]). In [10] the author initiated the study of a more general class of finite operators called generalized pair of finite operators defined by

$$\mathcal{GF} = \{(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H) : \|AX - XB - I\| \geq 1, \forall X \in \mathcal{B}(H)\}.$$

Now we will prove that the class (\mathcal{Y}) of operators is included in the class of finite operators. For this we need the following lemma.

Lemma 3.2. *If $S \in (\mathcal{Y})$, then $\sigma_{ar}(S) \neq \phi$.*

Proof. It is known that $\sigma_{ar}(S) \subset \sigma_a(S)$. Since $\sigma_a(S) \neq 0$, it suffices to prove that $\sigma_a(S) \subset \sigma_{ar}(S)$. If $S \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$\| |SS^* - S^*S|^{\frac{\alpha}{2}} x \| \leq k_\alpha^2 \| (S - \lambda I)x \| \text{ for all } x \in H \text{ and for all } \lambda \in \mathbb{C}. \quad (2.1)$$

Since

$$(S - \mu I)(S - \mu I)^* = SS^* - S^*S + (S - \mu I)^*(S - \mu I) \text{ for all } \mu \in \mathbb{C},$$

then

$$| \langle (SS^* - S^*S)x, x \rangle | \leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2, \text{ for all } x \in H.$$

Indeed, consider the polar decomposition of the operator $SS^* - S^*S = VD$, where $D = |SS^* - S^*S|$. Then V is a Hermitian partial isometry which commutes with D because $SS^* - S^*S$ is Hermitian. Hence, for any $x \in H$ such that $\|x\| = 1$

$$\begin{aligned} | \langle |SS^* - S^*S| x, x \rangle | &\leq \left| \langle |SS^* - S^*S|^{\frac{1}{2}} x, |SS^* - S^*S|^{\frac{1}{2}} V^* x \rangle \right| \\ &\leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\| \left\| |SS^* - S^*S|^{\frac{1}{2}} V^* x \right\| \\ &= \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\| \left\| V^* |SS^* - S^*S|^{\frac{1}{2}} x \right\| \\ &\leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2. \end{aligned}$$

Consequently

$$\| (S - \mu I)^* x \|^2 \leq \| (S - \mu I)x \|^2 + \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2, \quad (2.2)$$

for all $\mu \in \mathbb{C}$ and for all $x \in \mathcal{H}$. Let $\lambda \in \sigma_a(S)$, then there exists a normed sequence $(x_n)_n \subset \mathcal{H}$ such that $\| (S - \lambda I)x_n \| \rightarrow 0$. Therefore for $\lambda = \mu$, $x_n = x$, and for all n we get

$$\| |SS^* - S^*S|^{\frac{\alpha}{2}} x_n \| \leq k_\alpha^2 \| (S - \mu I)x_n \|. \quad (2.3)$$

By applying (2.2) and (2.3) we deduce that

$$\| (S - \mu I)^* x \|^2 \leq (1 + k_\alpha^2) \| (S - \mu I)x \|^2, \text{ for all } n.$$

Therefore $\| (S - \mu I)^* x \| \rightarrow 0$ and $\lambda \in \sigma_{ar}(S)$, that is, $\sigma_{ar}(S) \neq \phi$. \checkmark

Now we are ready to show that $(\mathcal{Y}) \subset \mathcal{F}$.

Theorem 3.2. *The class (\mathcal{Y}) of operators is included in the class of finite operators.*

Proof. It is shown in [9] that if $\sigma_a(A) \neq \phi$, then A is finite. It suffices to apply the above lemma. \checkmark

It is shown in [13] that:

- (1) If $T \in (\mathcal{Y})$ and $T^* \in (\mathcal{Y})$, then T is normal.
- (2) If $T \in (\mathcal{Y})$ is a compact operator, then T is normal.
- (3) if $T \in (\mathcal{Y})$ is similar to a normal operator, then T is normal.

In this section we will continue this study.

We begin by the following lemma. Note that concerning this lemma more general results in this direction can be found in [11] and [4]. Recall that an operator T is said to be algebraic if $P(T) = 0$ for a certain polynomial P .

Lemma 3.3. *If T is an algebraic operator, then $\sigma(T) = \sigma_p(T)$ (point spectrum of T).*

Proof. It is well known that an operator T is algebraic if and only if its spectrum consists of poles only. But a pole of an operator is always an eigenvalue. Hence for an algebraic operator the spectrum and the point spectrum coincide. \checkmark

In the following propositions, theorems and corollaries we will show other cases for the normality in (\mathcal{Y}) -classes.

Theorem 3.3. *Let $T \in (\mathcal{Y})$. If T is an algebraic operator, then T is normal.*

Proof. According to [13] the nilpotent operators of order n in (\mathcal{Y}) are null. Since $T \in (\mathcal{Y})$ implies the existence of a number $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \mu I)^* (T - \mu I), \forall \mu \in \mathbb{C},$$

it follows that for all $\mu \in \mathbb{C}$, that

$$|(T - \mu I)(T - \mu I)^* - (T - \mu I)^*(T - \mu I)|^\alpha = |TT^* - T^*T|^\alpha \leq 1$$

$$k_\alpha^2 (T - \mu I)^* (T - \mu I) \leq k_\alpha^2 [(T - \mu I) - (T - \mu I)]^* [(T - \mu I) - (T - \mu I)].$$

Therefore,

$$(T - \mu I)(T - \mu I)^* - (T - \mu I)^*(T - \mu I) \leq k_\alpha^2 [(T - \mu I) - \gamma I]^* [(T - \mu I) - \gamma I],$$

for all $\gamma \in \mathbb{C}$. Hence $(T - \mu I) \in (\mathcal{Y})$, for all $\mu \in \mathbb{C}$. By applying the above lemma we get $\sigma(T) = \sigma_p(T)$. Finally Theorem 3.1(i) ensures that T is normal. \checkmark

Theorem 3.4. *Let $T \in (\mathcal{Y})$. If there exists a polynomial P such that $P(T)$ is normal, then T is normal.*

Proof. According to [6] there exist reducing subspaces (H_n) for T such that $H = \bigoplus_n H_n$, where $T_0 = T|_{H_0}$ is algebraic and $T_n = T|_{H_n}$ ($n \geq 1$) is similar to a normal operator. Hence from Theorem 3.1 and [13] we complete the proof. \checkmark

4. Normality in (WN) -classes

Proposition 4.1. [3, Theorem 2.3] *Let $T \in (WN)$. If T is similar to a normal operator, then T is normal.*

Recall that an operator T is said to be isometric if $T^*T = I$ and co-isometric if $TT^* = I$.

Proposition 4.2. *Let $T \in (WN)$. If T is co-isometric, then T is unitary.*

Proof. It is known [5] that if an operator in (WN) is right invertible, then it is invertible. \square

Remark 4.1. *It is evident that any polynomial of a normal operator is normal, but the converse is not true. As an example take $T \in B(H)$ such that $T^2 = 0$. It is shown in [20] that:*

Theorem 4.1. [13] *Let $T \in (WN)$. If T^p and T^q are normal operators for certain coprime integers p, q , then T is normal.*

Remark 4.2. *The normality of T^2 for a certain operator $T \in (WN)$ it is not sufficient to ensure the normality of T . As an example take*

$$\dim H = 2, \quad T = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix},$$

$T \in (WN)$. It is clear that T is normal but

$$T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is not normal.

Theorem 4.2. *Let $T \in (WN)$. If T is a partial isometry and $0 \notin W(T)$, then T is normal.*

Proof. It is known [5] that If T is a partial isometry in (WN) , then T is quasinormal. Therefore

$$[T, T^*]T = 0.$$

If there exists a vector $x \in H$ such that $x \notin \ker[T, T^*]$, set

$$y = \frac{[T, T^*]x}{\|[T, T^*]\|},$$

hence $\langle Ty, y \rangle \in W(T)$. Since $0 \notin W(T)$, it results that $\langle Ty, y \rangle$ is not null. This contradicts the fact that $[T, T^*]T = 0$. Consequently $\ker[T, T^*] = H$, i.e., T is normal. \square

Now we are ready to prove that the conjecture holds for operators in (WN) .

Theorem 4.3. *If A or $A^* \in (WN)$, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

Proof. We start with the second assumption. Suppose that $A^* \in (WN)$ and $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$. Let $\lambda \in \sigma_p(T)$ such that $E = \ker(T - \lambda)$ be finite dimensional, then the subspace E is invariant under T and A^* . Since $A^* \in (WN)$, E reduces A^* . Let $H = E \oplus E^\perp$, hence we can write

$$A^* = \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix}, \quad T = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}.$$

Since $T \in \overline{R(\delta_A)}^w$, $\lambda I_E \in R(\delta_{A_1})$ and this implies that $\lambda = 0$. Since T is a compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$, it results that $\sigma(T) = \{0\}$ which implies that T is quasinilpotent. This completes the proof of the second assumption.

Remark that if $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then

$$T^* \in \overline{R(\delta_{A^*})}^w \cap \{A\}'.$$

Then the first assumption of the theorem follows in exactly the same way as the second. \checkmark

By the same arguments as in the above proof we prove the following theorem

Theorem 4.4. *If A or A^* is of class (\mathcal{Y}) , then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

Acknowledgment. I would like to thank the referee for his careful reading of the paper. His valuable suggestions and critical remarks made numerous improvements throughout. This work is supported, by the research center project No. Math/2005/04.

References

- [1] J. H. ANDERSON, On normal derivations, *Proc. Amer. Math. Soc.*, **38** (1973), 135–140.
- [2] S. K. BERBERIAN, An extension of Weyl's theorem to a class of not necessarily normal operators, *Michigan. Math. J.*, **16** (1969), 273–279
- [3] C. K. FONG AND V. I. ISTRATESCU, Some characterizations of hermitian operators and related classes of operators I, *Proc. Amer. Math. Soc.*, **76** (1979), 107–112.
- [4] M. A. KAASHOEK AND M. R. F. SMYTH, On operators T such that $f(T)$ is Riesz or meromorphic, *Proc. Royal Irish Acad.*, **72** (1972), 81–87.
- [5] F. KITTANEH, Linear operators for which $(\operatorname{Re}T)^2 \leq |T|^2$, *Tamkang J. Math.*, **11** no. 1 (1980), 11–115.
- [6] F. KITTANEH, On the structure of polynomially normal operators, *Bull. Austral. Math. Soc.*, **30** (1984), 11–18.
- [7] S. MECHERI, Some variants of Weber's theorem, *Mathematical Proceeding of the Royal Irish Academy*, **104A** no. 1 (2004), 67–73.
- [8] S. MECHERI, Commutants and derivation ranges, *Czechoslovak. Math. J.*, **49** no. 124 (1999), 843–847.
- [9] S. MECHERI, Finite operators, *Demonstratio. Math.*, **37** (2002), 357–366.
- [10] S. MECHERI, Generalized finite operators, *Demonstratio. Math.*, **38** no.1 (2005), 163–167.

- [11] C. OLSEN, A structure theorem for polynomial compact operators, *American. J. Math.*, **63** (1971), 686.
- [12] J. G. STAMPFLI AND B. L. WADHWA, On dominant operators, *Monatsh. Math.*, **84** (1977), 143–153.
- [13] A. UCHIYAMA AND T. YOSHINO, On the class (\mathcal{J}) operators, *Nihonkai. Math. J.*, **8** (1997), 179–194.
- [14] J. P. WILLIAMS, Finite operators, *Proc. Amer. Math. Soc.*, **26** (1970), 129–135.
- [15] R. E. WEBER, Derivation and the trace class operators, *Proc. Amer. Math. Soc.*, **73** (1979), 79–82.

(Recibido en **COMPLETAR** de 2005. Revisado en **COMPLETAR** de 2005)

DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCE
KING SAUD UNIVERSITY
P.O.BOX 2455, RIYADH 11451
SAUDI ARABIA
e-mail: mecherisalah@hotmail.com