Bull. Korean Math. Soc. **43** (2006), No. 4, pp. 747–753

# FUGLEDE-PUTNAM THEOREM FOR $p$-HYPONORMAL OR CLASS $\mathcal{Y}$ OPERATORS

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Abstract. We say operators $A, B$ on Hilbert space satisfy Fuglede-Putnam theorem if $AX = XB$ for some $X$ implies $A^*X = XB^*$. We show that if either (1) $A$ is $p$-hyponormal and $B^*$ is a class $\mathcal{Y}$ operator or (2) $A$ is a class $\mathcal{Y}$ operator and $B^*$ is $p$-hyponormal, then $A, B$ satisfy Fuglede-Putnam theorem.

## 1. Introduction

Our aim is to extend the Fuglede-Putnam theorem ([4], [7]). Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the algebras of all bounded linear operators on $\mathcal{H}, \mathcal{K}$. The familiar Fuglede-Putnam theorem is as follows:

Theorem 1 (Fuglede-Putnam [4], [7]). *If $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ be normal and $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Many authors have extented this theorem for several classes of operators, for examples [3], [5], [6], [10], [13], [15], [17]. We say operators $A, B$ satisfy Fuglede-Putnam theorem if $AX = XB$ implies $A^*X = XB^*$. The aim of this paper is to show that if either (1) $A$ is $p$-hyponormal and $B^*$ is a class $\mathcal{Y}$ operator or (2) $A$ is a class $\mathcal{Y}$ operator and $B^*$ is $p$-hyponormal, then $A, B$ satisfy Fuglede-Putnam theorem. We remark that B. P. Duggal [3] proved if $A, B^*$ are $p$-hyponormal operators, then $A, B$ satisfy Fugled-Putnam theorem, and A. Uchiyama and T. Yoshino [15] proved if $A, B^*$ are class $\mathcal{Y}$ operators, then $A, B$ satisfy Fugled-Putnam theorem.

Received June 21, 2005.
2000 Mathematics Subject Classification: 47B20.
Key words and phrases: $p$-hyponormal operator, class $\mathcal{Y}$, Fuglede-Putnam theorem.
[1] This research was supported by research center Project No. 2005-04.
[2] This research was supported by Grant-in-Aid Research No. 15540180.

An operator $A \in B(\mathcal{H})$ is said to be $p$-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where $p > 0$. This definition is due to Aluthge [1] and many authors studied interesting properties of $p$-hyponormal operators by using Aluthge transform (see [1], [6]). $A$ is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 1$ (or $A \in \mathcal{Y}_\alpha$) if there exists a positive number $k_\alpha$ such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A-\lambda)^*(A-\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator $A$ is $M$-hyponormal, i.e., there exists a positive number $M$ such that

$$(A-\lambda)(A-\lambda)^* \leq M^2 (A-\lambda)^*(A-\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C},$$

and $M$-hyponormal operators are class $\mathcal{Y}_2$ operators (see [15]). $A$ is said to be dominant if for any $\lambda \in \mathbb{C}$ there exists a positive number $M_\lambda$ such that

$$(A-\lambda)(A-\lambda)^* \leq M_\lambda^2 (A-\lambda)^*(A-\lambda).$$

It is obvious that $M$-hyponormal operators are dominant, but the converse does not hold. Let $\{f_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Define $Tf_n = 2^{-|n|} f_{n+1}$. It is known that $T$ is a dominant operator which is not a class $\mathcal{Y}$ operator. (Hence $T$ is not $M$-hyponormal.) We remark $T$ is not $p$-hyponormal, as $\langle (T^*T)^p f_1, f_1 \rangle = 4^{-p} < 1 = \langle (TT^*)^p f_1, f_1 \rangle$ (see [11], [15]). Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define $Sf_1 = f_2, Sf_2 = 2f_3, Sf_n = f_{n+1}$ for $n = 3, 4, \ldots$. Wadhwa [16] proved $S$ is $M$-hyponormal, hence $S$ is a class $\mathcal{Y}$ operator. But $S$ is not $p$-hyponormal for any $0 < p$, as $\langle (S^*S)^p f_3, f_3 \rangle = 1 < 2^p = \langle (SS^*)^p f_3, f_3 \rangle$. However it is not known that there exists a $p$-hyponormal operator which is not a class $\mathcal{Y}$ operator. Also, it is not known that there exists a class $\mathcal{Y}$ operator which is not dominant.

## 2. Results

We will recall some known results which will be used in the sequel.

Lemma 2. (Uchiyama and Yoshino [15]) *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under $A$. If $A|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces $A$.*

Lemma 3 (Uchiyama [14]). *Let $A \in B(\mathcal{H})$ be $p$-hyponormal and $\mathcal{M} \subset \mathcal{H}$ be invariant under $A$. If $A|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces $A$.*

LEMMA 4 (Stampfli and Wadhwa [11]). *Let $A \in B(\mathcal{H})$ be dominant. Let $\delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(A - \lambda)f(\lambda) = x \neq 0$ for some $x \in \mathcal{H}$, then there exists an analytic function $g : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(A - \lambda)g(\lambda) = x$.*

REMARK. In [11], the authors assert $f$ is analytic. But they use Putnam's result [9], i.e., if $A = \int \lambda dE(\lambda)$ is normal, then

$$\bigcap \{(A - \lambda)\mathcal{H} | \lambda \in \mathbb{C} \setminus \delta\} = E(\delta)\mathcal{H}$$
$$= \{x \in \mathcal{H} | \exists \text{ analytic } g : \mathbb{C} \setminus \delta \to \mathcal{H} \text{ such that } (A - \lambda)g(\lambda) = x\}.$$

Hence we must substitute a bounded function $f$ by an analytic function $g$. If $A$ is pure, i.e., $A$ has no-nonzero reducing subspace $\mathcal{M}$ such that $A|_{\mathcal{M}}$ is normal, then $\ker A = \{0\}$ as $\ker A \subset \ker A^*$. Hence $f = g$. This is pointed by Professor F. Hiai.

The following result is due to Takahashi [12]. We denote by [ran $A$] the closure of the range of $A$.

LEMMA 5 (Takahashi [12]). *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then the following assertions are equivalent.*

*(1) $A, B$ satisfy Fugled-Putnam theorem.*

*(2) If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then [ran $C$] reduces $A$, $(\ker C)^\perp$ reduces $B$, and $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are normal.*

REMARK. In (2), $C_1 : (\ker C)^\perp \ni x \to Cx \in [\text{ran } C]$ is a quasi-affinity (i.e., $C_1$ is injective and has dense range) such that $A|_{[\text{ran } C]}C_1 = C_1 B|_{(\ker C)^\perp}$. Then $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are unitarily equivalent normal operators by a corollary of the Fuglede-Putnam theorem (see Theorem 1.6.4 of [8] and its proof).

LEMMA 6. *Let $A \in B(\mathcal{H})$ be an injective $p$-hyponormal operator and $B^* \in B(\mathcal{K})$ be a class $\mathcal{Y}$ operator. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover, [ran $C$] reduces $A$, $(\ker C)^\perp$ reduces $B$, and $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are unitarily equivalent normal operators.*

*Proof.* (Case $1/2 \leq p \leq 1$) Since $B^*$ is class $\mathcal{Y}$, there exist positive numbers $\alpha$ and $k_\alpha$ such that

$$|BB^* - B^*B|^\alpha \leq k_\alpha^2 (B - \lambda)(B - \lambda)^* \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence for $x \in |BB^* - B^*B|^{\alpha/2}\mathcal{K}$ there exists a bounded function $f : \mathbb{C} \to \mathcal{K}$ such that

$$(B - \lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in \mathbb{C}$$

by [2]. Let $A = U|A|$ be the polar decomposition of $A$ and define its Aluthge transform by $\tilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\tilde{A}$ is hyponormal by [1] (the author assumed $U$ is unitary, however this assumption is not necessary.) Then

$$\begin{aligned}(\tilde{A}-\lambda)|A|^{1/2}Cf(\lambda) &= |A|^{1/2}(A-\lambda)Cf(\lambda)\\ &= |A|^{1/2}C(B-\lambda)f(\lambda) = |A|^{1/2}Cx\end{aligned}$$

for all $\lambda \in \mathbb{C}$.

We assert $|A|^{1/2}Cx = 0$. Because if $|A|^{1/2}Cx \neq 0$, there exists an analytic function $g : \mathbb{C} \to \mathcal{H}$ such that $(\tilde{A}-\lambda)g(\lambda) = |A|^{1/2}Cx$ by Lemma 4. Since

$$g(\lambda) = (\tilde{A}-\lambda)^{-1}|A|^{1/2}Cx \to 0 \text{ as } \lambda \to \infty,$$

we have $g(\lambda) = 0$, and hence $|A|^{1/2}Cx = 0$. This is a contradiction.

Then

$$|A|^{1/2}C|BB^* - B^*B|^{\alpha/2}\mathcal{K} = \{0\}.$$

Since $\ker A = \ker |A| = \{0\}$, we have

$$C(BB^* - B^*B) = 0.$$

Since [ran $C$] is invariant under $A$ and $(\ker C)^{\perp}$ is invariant under $B^*$, we can write

$$A = \begin{pmatrix} A_1 & S \\ 0 & A_2 \end{pmatrix} \text{ on } \mathcal{H} = [\text{ran } C] \oplus [\text{ran } C]^{\perp},$$

$$B = \begin{pmatrix} B_1 & 0 \\ T & B_2 \end{pmatrix} \text{ on } \mathcal{K} = (\ker C)^{\perp} \oplus (\ker C),$$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker C)^{\perp} \oplus (\ker C) \to [\text{ran } C] \oplus [\text{ran } C]^{\perp}.$$

Then

$$\begin{aligned}0 &= C(BB^* - B^*B)\\ &= \begin{pmatrix} C_1(B_1B_1^* - B_1^*B_1 - T^*T) & C_1(B_1T^* - T^*B_2) \\ 0 & 0 \end{pmatrix}\end{aligned}$$

and

$$C_1(B_1B_1^* - B_1^*B_1 - T^*T) = 0.$$

Since $C_1$ is injective and has dense range,

$$B_1B_1^* - B_1^*B_1 - T^*T = 0$$

and

$$B_1B_1^* = B_1^*B_1 + T^*T \geq B_1^*B_1.$$

This implies $B_1^*$ is hyponormal. Since $AC = CB$, we have

$$A_1C_1 = C_1B_1$$

where $A_1$ is $p$-hyponormal by [14]. Hence $A_1, B_1$ are normal and

$$A_1^*C_1 = C_1B_1^*$$

by [3]. Then $S = 0$ by Lemma 3 and $T = 0$ by Lemma 2. Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1B_1^* & 0 \\ 0 & 0 \end{pmatrix} = CB^*.$$

Hence $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are normal by Lemma 5 and unitarily equivalent by its remark.

(Case $0 < p < 1/2$) Let $A = U|A|$ be the polar decomposition of $A$ and define its Aluthge transform by $\tilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\tilde{A}$ is $(p + 1/2)$-hyponormal by [1] and

$$\tilde{A}|A|^{1/2}C = |A|^{1/2}AC = |A|^{1/2}CB.$$

Let $\tilde{A} = V|\tilde{A}|$ be the polar decomposition and $\hat{A} = |\tilde{A}|^{1/2}V|\tilde{A}|^{1/2}$. Then $\hat{A}$ is hyponormal and

$$\hat{A}|\tilde{A}|^{1/2}|A|^{1/2}C = |\tilde{A}|^{1/2}|A|^{1/2}CB.$$

Since $\sigma_p(\tilde{A}) = \sigma_p(A) = \emptyset$, we have $C(BB^* - B^*B) = 0$ by an similar arguments in the case $1/2 \leq p \leq 1$. The rest is the same to the case $1/2 \leq p \leq 1$. □

THEOREM 7. *Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$. If either* (1) *$A$ is $p$-hyponormal and $B^*$ is a class $\mathcal{Y}$ operator or* (2) *$A$ is a class $\mathcal{Y}$ operator and $B^*$ is $p$-hyponormal, then $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$ implies $A^*C = CB^*$. Moreover,* $[\text{ran } C]$ *reduces $A$,* $(\ker C)^\perp$ *reduces $B$, and $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are unitarily equivalent normal operators.*

*Proof.* (1). Decompose $A$ into normal part $A_1$ and pure part $A_2$ as

$$A = A_1 \oplus A_2 \quad \text{on } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and write

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{K} \to \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since $\ker A_2 \subset \ker A_2^*$ and $A_2$ is pure, $A_2$ is injective. $AC = CB$ implies

$$\begin{pmatrix} A_1C_1 \\ A_2C_2 \end{pmatrix} = \begin{pmatrix} C_1B \\ C_2B \end{pmatrix}.$$

Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 \\ A_2^*C_2 \end{pmatrix} = \begin{pmatrix} C_1B^* \\ C_2B^* \end{pmatrix} = CB^*$$

by [3] and Lemma 6. The rest follows from Lemma 5 and its remark.

(2). Since $AC = CB$, we have $B^*C^* = C^*A^*$. Hence $BC^* = B^{**}C^* = C^*A^{**} = C^*A$ by (1) and $A^*C = CB^*$. The rest follows from Lemma 5 and its remark. □

Acknowledgement. The authors would like to express their thanks to the referee for his helpful comments.

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