13) ($\Rightarrow$) Assume that $\mathcal{A} \subseteq \mathcal{F}$. Then $\sigma(\mathcal{A})$ is the intersection of all σ-fields containing $\mathcal{A}$ and so is contained in any such σ-field. In particular, $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

($\Leftarrow$) Assume that $\sigma(\mathcal{A}) \subseteq \mathcal{F}$. Trivially $\mathcal{A} \subseteq \sigma(\mathcal{A})$ and so $\mathcal{A} \subseteq \mathcal{F}$.

14) Let $\mathcal{B} = \mathcal{B}((a,b])$ be the σ-field generated by $\mathcal{P}$, known as the Borel sets of $\mathbb{R}$.

(i) $\mathcal{B}([a,b]) = \mathcal{B}((a,b])$.

Note that

$$[a,b] = \bigcap_{n\geq 1} \left(a - \frac{1}{n}, b\right] \in \mathcal{B}((a,b])$$

since a σ-field is closed under countable intersections. So $\mathcal{B}((a,b])$ is **a** σ-field containing all $[a,b]$ while $\mathcal{B}([a,b])$ is **the** smallest such σ-field. Hence $\mathcal{B}([a,b]) \subseteq \mathcal{B}((a,b])$. Similarly

$$(a,b] = \bigcup_{n\geq 1} \left[a + \frac{1}{n}, b\right] \in \mathcal{B}([a,b]),$$

giving $\mathcal{B}((a,b]) \subseteq \mathcal{B}([a,b])$.

Hence $\mathcal{B}([a,b]) = \mathcal{B}((a,b])$.

(ii) To prove $\mathcal{B}([a,b)) = \mathcal{B}((a,b])$ it suffices, by part (i) to prove that $\mathcal{B}([a,b)) = \mathcal{B}([a,b])$. This follows as in (i) from the two equalities

$$[a,b) = \bigcup_{n\geq 1} \left[a, b - \frac{1}{n}\right]$$

and

$$[a,b] = \bigcap_{n\geq 1} \left[a, b + \frac{1}{n}\right).$$

(iii) You might be happy with $\{x\} = [x,x] \in \mathcal{B}$ by (ii) or you could write

$$\{x\} = \bigcap_{n\geq 1} \left(x - \frac{1}{n}, x + \frac{1}{n}\right] \in \mathcal{B}.$$

(iv)

$$\mathbb{Q} = \bigcup_{r\in\mathbb{Q}} \{r\}$$

a countable union of sets that by (iii) are in $\mathcal{B}$. Hence $\mathbb{Q} \in \mathcal{B}$.

(v) Recall that σ-fields are closed under complements so

$$\{\text{irrationals}\} = \mathbb{Q}^c \in \mathcal{B}.$$

(vi) Let $A \in$ co-finite topology. Then either $A = \phi \in \mathcal{B}$ or A^c is finite. In the second case we can write

$$\begin{aligned} A^c &= \{x_1, x_2, ..., x_r\} \\ &= \bigcup_{i=1}^{r} \{x_i\} \in \mathcal{B}. \end{aligned}$$

Thus $A = (A^c)^c \in \mathcal{B}$.

Hence the co-finite topology on $\mathbb{R}$ is a subset of $\mathcal{B}$.

15) At every point x_0 of discontinuity we have

$$\lim_{\substack{x \to x_0 \\ x < x_0}} F(x) < \lim_{\substack{x \to x_0 \\ x > x_0}} F(x) = F(x_0).$$

From the first year we know that between any two real numbers we can find a rational, so we can find a rational $r = r(x_0)$ satisfying

$$\lim_{\substack{x \to x_0 \\ x < x_0}} F(x) < r < \lim_{\substack{x \to x_0 \\ x > x_0}} F(x) = F(x_0).$$

Now if x_1 is a point of discontinuity which larger than x_0 then

$$\begin{aligned} r(x_0) &< F(x_0) \leq \lim_{\substack{x \to x_1 \\ x < x_1}} F(x) \qquad \text{since } F \text{ is monotonic increasing,} \\ &< r(x_1) < \lim_{\substack{x \to x_1 \\ x > x_1}} F(x). \end{aligned}$$

So the sequence of rationals we choose, $r(x_i)$, are distinct. The collection of all rationals is countable so the collection of discontinuities must be countable.

16) If $y > x$ then

$$F(y) - F(x) = \sum_{x < n \leq y} p_n \geq 0$$

since the $p_n \geq 0$. So F is increasing. Also

$$\lim_{\substack{y \to x \\ y > x}} F(y) - F(x) = \lim_{\substack{y \to x \\ y > x}} \sum_{x < n \leq y} p_n.$$

But if y is sufficiently close to x then $(x, y]$ never contains an integer. To see this assume first that x is an integer, when if $x < y < x+1$ then $(x, y]$ does not contain an integer. Otherwise x is not an integer. But if n is the smallest integer $x < n$ then if $y < n$ also we see again that $(x, y]$ does not contain an integer. So $\sum_{x<n\leq y} p_n = 0$ if y is sufficiently is close to x. Thus

$$\lim_{\substack{y \to x \\ y > x}} F(y) = F(x),$$

and so F is right continuous. Thus F is a distribution function.

If we had $n < x$ instead of $n \leq x$ in the definition of $F(x)$ then in the above argument we would have a sum over integers in $[x, y)$. If x were an integer then this interval would always contain integers however close y was to x. Thus $F(x)$ would not necessarily be right continuous. (But what about being left continuous?)

17) To check that μ is additive we need to verify that if given any collection of disjoint sets $\{A_i\}_{1\leq i\leq N} \subseteq \mathcal{C}$ such that $\bigcup_{i=1}^{N} A_i \in \mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu(A_i).$$

But the condition here is satisfied in the present example only when $A_1 = [0, 1/4)$ and $A_2 = [1/4, 3/4)$ for then $A_1 \cup A_2 = [0, 3/4) \in \mathcal{C}$. So we check that

$$\mu(A_1 \cup A_2) = \mu([0, 3/4)) = 4$$

while

$$\mu(A_1) + \mu(A_2) = \mu([0, 1/4)) + \mu([1/4, 3/4)) = 2 + 2 = 4.$$

Equality means that μ is additive on $\mathcal{C}$.

The ring generated by $\mathcal{C}$ must be closed under unions and intersections. So we start with $\mathcal{C}$ and add in the sets formed by taking unions and differences. This leads to

ϕ,	X,	$[0, 1/4)$,	$[0, 1/2)$,
$[0, 3/4)$,	$[1/4, 3/4)$,	$[1/4, 1/2)$,	$[1/2, 3/4)$,
$[1/4, 1)$,	$[1/2, 1)$,	$[3/4, 1)$,	$[0, 1/4) \cup [3/4, 1)$,
$[0, 1/2) \cup [3/4, 1)$,	$[0, 1/4) \cup [1/2, 3/4)$,	$[0, 1/4) \cup [1/2, 1)$,	$[1/4, 1/2) \cup [3/4, 1)$.

This is the **smallest** collection you can make from $\mathcal{C}$ by adding in unions and differences. You should check that this **is** a ring in which case it is the smallest ring containing $\mathcal{C}$ and so the ring generated by $\mathcal{C}$.

In fact the ring consists of all possible unions of the four intervals

$$[0,1/4), \quad [1/4,1/2), \quad [1/2,3/4), \quad [3/4,1).$$

In particular we see that our ring will contain 2^4 elements.

If we can extend μ to the ring it should take values on these four intervals. It is not hard to see from the information given that we must have

$$\mu([0,1/4)) = 2, \quad \mu([1/4,1/2)) = 0, \quad \mu([1/2,3/4)) = 2, \quad \mu([3/4,1)) = 0.$$

Then by additivity every interval in our ring has a measure.

18) To be a semi-ring $\mathcal{C}$ has to be closed under intersections while the difference of two sets from $\mathcal{C}$ should be the union of sets from $\mathcal{C}$. By observation the intersection of any two sets from $\mathcal{C}$ lies in $\mathcal{C}$. The only non-trivial difference we can take is

$$X \setminus \{2,3\} = \{1\} \cup \{4,5\},$$

a union of sets from $\mathcal{C}$ as required. (I say that all other differences trivial in that they lie in $\mathcal{C}$.)

To show that μ is additive we need to verify the definition repeated in question 17. In this example there are three collections $\{A_i\} \subseteq \mathcal{C}$ with $\bigcup_i A_i \in \mathcal{C}$. This means that we have to check the following three equalities.

$$\begin{aligned} \mu(\{1\}) + \mu(\{2,3\}) &= \mu(\{1,2,3\}), && \text{(a)} \\ \mu(\{1\}) + \mu(\{2,3\}) + \mu(\{4,5\}) &= \mu(X), && \text{(b)} \\ \mu(\{1,2,3\}) + \mu(\{4,5\}) &= \mu(X). && \text{(c)} \end{aligned}$$

Yet $LHS(a) = 1 + 1 = 2$ while $RHS(a) = 2$ so (a) holds. Similarly $LHS(b) = 1 + 1 + 1 = 3$ while $RHS(b) = 3$ so (b) holds. Finally, (c) follows from (a) and (b). Hence μ is additive.

As in the last question we add in all unions and differences of the intervals in $\mathcal{C}$ to get the ring generated by $\mathcal{C}$. Note that in taking unions and differences we never would expect to split up the pair $\{2,3\}$ nor the pair $\{4,5\}$. So we might expect the ring to contain all subsets of $\{1, \{2,3\}, \{4,5\}\}$, a set of **three** elements, and so the ring would contain $2^3 = 8$ sets. In fact we find the ring consists of

$$\begin{array}{llll} \phi, & X, & \{2,3\}, & \{1\}, \\ \{4,5\}, & \{1,2,3\}, & \{1\} \cup \{4,5\}, & \{2,3\} \cup \{4,5\}. \end{array}$$

We extend μ to $\mathcal{R}$ by defining

$$\mu(\{1\} \cup \{4,5\}) = 2, \quad \mu(\{2,3\} \cup \{4,5\}) = 2.$$

So μ is a non-negative and additive. Since X is finite this trivially means σ-additive and so μ is a measure.

19) In all cases we need to check the three conditions for λ to be an outer measure.

1. $\lambda(\phi) = 0$
2. If $E \subseteq F$ then $\lambda(E) \leq \lambda(F)$, (Monotonic)
3. $\lambda\left(\bigcup_1^\infty A_i\right) \leq \sum_1^\infty \lambda(A_i)$, (countably subadditive).

Note that 1. holds in all three examples so we need only check 2. and 3.

(a) 2. Assume $E \subseteq F$.

If $F = \phi$ then necessarily $E = \phi$ and so $\lambda(E) = 0 = \lambda(F)$.

If $F \neq \phi$ then $\lambda(F) = 1$ which is greater than or equal to any value (i.e. 0 or 1) that $\lambda(E)$ can take.

Hence, in all cases, $\lambda(E) \leq \lambda(F)$.

3. Let $\{A_i\}_{i\geq 1}$ be given. If $A_i = \phi$ for all $i \geq 1$ then $\bigcup_1^\infty A_i = \phi$ and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 0 = \sum_1^\infty \lambda(A_i).$$

Otherwise there exists m such that $A_m \neq \phi$. Then $\bigcup_1^\infty A_i \neq \phi$ and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 1 = \lambda(A_m) \leq \sum_1^\infty \lambda(A_i)$$

since $\lambda \geq 0$.

Hence λ is an outer measure.

The λ-measurable sets satisfy

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \tag{1}$$

for every $A \subseteq X$. So for E to be λ-measurable we need it to satisfy

$$1 = \lambda(E) + \lambda(E^c),$$

having put $A = X$ in (1). This can only be satisfied if either $E = \phi$ and $E^c \neq \phi$, or $E^c = \phi$ and $E \neq \phi$. That is, if either $E = \phi$ or $E = X$. Remember these are only **possible** λ-measurable sets since (1) should hold for all A not just $A = X$. But we can check that ϕ and X **are** λ-measurable. Yet ϕ and X are **always** λ-measurable whatever the problem To see this simply observe that $E = \phi$ in (1) gives $\lambda(A) = 0 + \lambda(A)$ which is true for all A while $E = X$ in (1) gives $\lambda(A) = \lambda(A) + 0$ which again is true for all A.

So the λ-measurable sets are ϕ and X.

b) 2. Assume $E \subseteq F$.

If $E = \phi$ then $\lambda(E) = 0$ which is less than or equal to any value (0,1 or 2) that can be taken by $\lambda(F)$, so $\lambda(E) \leq \lambda(F)$.

If $\phi \neq E \neq X$ then $\lambda(E) = 1$ but also F is necessarily not empty. So $\lambda(F) \geq 1 = \lambda(E)$.

If $E = X$ then necessarily $F = X$ and so $\lambda(F) = \lambda(E)$.

Hence, in all cases, $\lambda(E) \leq \lambda(F)$.

3. Let $\{A_i\}_{i\geq 1}$ be given. If $A_i = \phi$ for all $i \geq 1$ then $\bigcup_1^\infty A_i = \phi$ and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 0 = \sum_1^\infty \lambda(A_i).$$

If there exists m such that $A_m \neq \phi$ and $\bigcup_1^\infty A_i \neq X$ then $A_m \neq X$ and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 1 = \lambda\left(A_m\right) \leq \sum_1^\infty \lambda(A_i)$$

since $\lambda \geq 0$. If $A_m \neq \phi$ and $\bigcup_1^\infty A_i = X$ then either $A_m = X$ or $A_m \neq X$ and there exists $k \neq m$ with $A_k \neq \phi$. In the first case

$$\lambda\left(\bigcup_1^\infty A_i\right) = 2 = \lambda(A_m) \leq \sum_1^\infty \lambda(A_i)$$

while in the second case

$$\lambda\left(\bigcup_1^\infty A_i\right) = 2 \leq \lambda(A_m) + \lambda(A_k) \leq \sum_1^\infty \lambda(A_i).$$

Hence in all cases λ is sub-additive. Hence λ is an outer measure.

As seen in part (a) the sets ϕ and X are λ-measurable.

Claim There are no other λ-measurable sets.

Proof Let $\phi \neq E \neq X$. So there exist $x \in E$ and $y \notin E$. Take as a test set $A = \{x, y\}$ in (1) above. Then $\lambda(A) = 1$ while $\lambda(A \cap E) + \lambda(A \cap E^c) = \lambda(\{x\}) + \lambda(\{y\}) = 1 + 1 = 2$. So we do not have equality and E is not λ-measurable.

c) 2. Assume $E \subseteq F$.

If E is countable then $\lambda(E) = 0$ which is less than or equal to any value (0 or 1) that can be taken by $\lambda(F)$, so $\lambda(E) \le \lambda(F)$.

If E is uncountable then F is uncountable so $\lambda(E) = 1 = \lambda(F)$.

Hence, in all cases $\lambda(E) \le \lambda(F)$.

3. Let $\{A_i\}_{i\ge1}$ be given. If A_i countable for all $i \ge 1$ then $\bigcup_1^\infty A_i$ is countable and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 0 = \sum_1^\infty \lambda(A_i).$$

Otherwise there exists m such that A_m is uncountable. Then $\bigcup_1^\infty A_i$ is also uncountable and so

$$\lambda\left(\bigcup_1^\infty A_i\right) = 1 = \lambda\left(A_m\right) \le \sum_1^\infty \lambda(A_i)$$

since $\lambda \ge 0$. In all cases $\lambda\left(\bigcup_1^\infty A_i\right) \le \sum_1^\infty \lambda(A_i)$. Thus λ is an outer measure.

As in part (a), for E to be λ-measurable we need it to satisfy

$$1 = \lambda(E) + \lambda(E^c),$$

having put $A = X$ in (1). This can only be satisfied if either E uncountable and E^c countable, or E^c countable and E uncountable. Since X is uncountable these are the same as either E^c countable or E countable. Remember these are only **possible** λ-measurable sets since (1) should hold for all A not just $A = X$. So we need to check that such sets are λ- measurable.

So, for example, let E be countable.

If the test set A is countable then both sides of (1) are 0 and we have equality.

Assume that the test set A is uncountable. Write $A = (A\cap E^c)\cup(A\cap E)$ and note that $A \cap E$ is countable since it is a subset of a countable set E. Thus A uncountable implies that $A\cap E^c$ is uncountable. Thus $\lambda(A\cap E^c) = 1$ in (1). As noted $A \cap E \subseteq E$ is countable and so $\lambda(A \cap E) = 0$. Finally $\lambda(A) = 1$ and so we have equality in (1). Thus (1) holds for all test sets A and so countable E are λ-measurable.

The proof for E^c uncountable is similar.

Hence the λ-measurable sets are either E^c countable or E countable.

20) Recall the definition of being λ-measurable as

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \qquad (1)$$

for every $A \subseteq X$. If we put $A = X$ we are asking that $1 = \lambda(E) + \lambda(E^c)$ for any λ-measurable set. Since $E \subseteq \mathbb{N}$, and N is infinite, one of E or E^c is infinite, i.e. one of $\lambda(E)$ or $\lambda(E^c) = 1$. Thus we need one of the two cases, $\lambda(E) = 1$ and $\lambda(E^c) = 0$ or $\lambda(E) = 0$ and $\lambda(E^c) = 1$. But $\lambda(E^c) = 0$ implies $E^c = \phi$, in which case $E = \mathbb{N}$, while $\lambda(E) = 0$ implies that $E = \phi$. So the only possible λ-measurable sets are $\mathbb{N}$ and ϕ. As noted in the last two questions the whole space and the empty set are always λ-measurable sets.