WEYL'S TYPE THEOREMS FOR POSINORMAL OPERATORS

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Abstract. Let A be a bounded linear operator acting on infinite dimensional separable Hilbert space H. The study of operators satisfying Weyl's theorem, Browder's theorm, the SVEP and Bishop's property is of significant interest, and is currently being done by a number of mathematicians around the world. It is known that Weyl's theorem holds for M-hyponormal operators, but does not hold for dominant operators. Hence it is an interesting problem to seek a condition which implies Weyl's theorem for dominant operators. In Ho Jeon *et at* [] proved that if A is dominant and satisfies $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M$ for every $M \in Lat(A)$, then Weyl's theorem holds for A. Recently X.Cao [] showed that generalized a-Weyl's theorem holds for $f(A)$, where f is an analytic function defined on an open neighborhood of $\sigma(A)$ in the case where A^* is p-hyponormal or M-hyponormal. Also Aiena [] showed that a-Weyl's theorem holds for some classes of operators. In this paper we prove that if A^* is conditionally totally posinormal (with certain condition) or totally posinormal, then generalized a-Weyl's theorem holds for A and for $f(A)$, where f is an analytic function defined on an open neighborhood of $\sigma(A)$.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := \dim N(A)$, $\beta(A) := \dim(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively.

An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension.

The index of a Fredholm operator is given by

$$\text{ind}I(A) = \alpha(A) - \beta(A).$$

An operator $A \in B(H)$ is called Weyl if it is a Fredholm of index zero, and Browder if it is Fredholm of finite ascent and descent, equivalently

[1] This research was supported by research center Project No. 2008-04

1991 *Mathematics Subject Classification.* 47B20.

Key words and phrases. C^*-algebra, Genaralized finite operator.

[?, Theorem 7.9.3] if A is Fredholm and $A-\lambda$ is invertible for sufficiently small $|\lambda|>0$, $\lambda\in\mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [?, ?]

$$\sigma_e(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Weyl}\},$$
$$\sigma_w(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Fredholm}\},$$
$$\sigma_b(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e A\subseteq\sigma_w(A)\subseteq\sigma_b A=\sigma_e(A)\cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K\subseteq\mathbb{C}$. If we write $isoK=K\setminus accK$, then we let

$$\pi_{00}(A):=\{\lambda\in iso\sigma(A):\ 0<\alpha(A-\lambda)<\infty\},$$
$$p_{00}(A):=\sigma(A)\setminus\sigma_b(A).$$

Definition 1. We say that Weyl's theorem holds for A if

$$\sigma(A)\setminus\sigma_w(A)=\pi_{00}(A).$$

Definition 2. We that the generalized Weyl's theorem holds for A if

$$\sigma(A)\setminus\sigma_{Bw}(A)=E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$ denote the isolated point of the spectrum which are eigenvalues (no restriction multiplicity) and the set of all complex numbers λ for which $A-\lambda I$ is not B-Weyl, respectively.

Let $A\in B(H)$, n be a nonnegative integer and define $A_{[n]}$ to be the restriction A to $R(A^n)$ viewed as a map from $R(A^n)$ to $R(A^n)$ (in particular $A_{[0]}=A$). If for some integer n, the range space $R(A^n)$ is closed and $A_{[n]}$ is upper (resp. a lower) semi Fredholm operator, then A is called an upper (resp. a lower) semi-B-Fredholm operator. More over if $A_{[n]}$ is a Fredholm (Weyl or Browder) operator , then A is called a B-Fredholm's spectrum $\sigma_{BF}(A)$, B-Weyl's spectrum $\sigma_{Bw}(A)$ and B-Browder's spectrum $\sigma_{BB}(A)$. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

Note that, if the generalized Weyl's theorem holds for A, then so does Weyl's theorem []. We say that Browder's theorem holds for A if

$$\sigma(A)\setminus\sigma_w(A)=p_{00}(A).$$

An operator $A\in B(H)$ is said to be posinormal (the word posinormal stands for positive normal), if there exists a $P\geq 0$ in $B(H)$ such that $AA^*=A^*PA$. Or equivalently, $A\in B(H)$ is posinormal if there exists a co-isometry $V^*\in B(H)$ and a positive operator $P\in B(H)$ such that $A=A^*PV^*$.

Rhaly [] introduced posinormal operators and proved and many interesting properties of posinormal operators, and has since considered by Jeon *et al* [].

The class of posinormal operators contains in particular, the classes consisting of hyponormal operators ($A \in B(H) : AA^* \leq A^*A$), M-hyponormal ($A \in B(H) : |(A-\lambda I)^*|^2 \leq M|(A-\lambda I)|^2$ for some real number $M > 0$) and dominant operators ($A \in B(H) : |(A-\lambda I)^*|^2 \leq M_\lambda |A-\lambda I|^2$ for some real number $M_\lambda > 0$ and all complex number λ). A posinormal operator A is said to be conditionally totally posinormal (resp., totally posinormal), shortened to $A \in CTP$ (resp., $A \in TP$), if to each complex number λ there corresponds a positive P_λ such that $|(A-\lambda I)^*|^2 = |P_\lambda^{\frac{1}{2}}|^2$ (resp., if there exists a positive operator P such that $|(A-\lambda I)^*|^2 = |P^{\frac{1}{2}}|^2$ for all λ).

In [], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian to operators to hyponormal and Toeplitz operators [], and to several classes of operators including semi-normal operators []. Berkani [] showed that if A is hyponormal, then A satisfies generalzed Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem. In this paper we show that if A is conditionally totally posinormal or totally posinormal with additional conditions, then generalized Weyl's theorem holds for $f(A)$, where f is an analytic function defined on an open neighborhood of $\sigma(A)$. Recently X.Cao [] showed that generalized a- Weyl's theorem holds for $f(A)$, where f is an analytic function defined on an open neighborhood of $\sigma(A)$ in the case where A^* is p-hyponormal or M-hyponormal. Also Aiena [] showed that a-Weyl's theorem holds for some classes of operators. In this paper we prove that if A^* is conditionally totally posinormal (with certain condition) or totally posinormal, then generalized a-Weyl's theorem holds for A and for $f(A)$, where f is an analytic function defined on an open neighborhood of $\sigma(A)$.

2. Main results

An operator $A \in B(H)$ is said to have Bishop's property (β) if $(A-z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D, then $f_n(z) \to 0$ uniformly on every compact subset of D. A is said to have the single valued extension property if there exists no nonzero analytic function f such that $(A-z)f(z) \equiv 0$. It is clear that if A has Bishop's property (β), then A has the single valued extension property. In this case, the local resolvent $\rho_A(x)$ of $x \in H$ denotes the maximal open set on which there exists unique analytic function $f(z)$ satisfying $(A-z)f(z) \equiv x$. The local spectrum $\sigma_A(x)$ of $x \in H$ is defined by $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$ and $X_A(F) = \{x \in H : \sigma_A(x) \subset H\}$ for a subset $F \subset \mathbb{C}$. A is said to have finite ascent if $\ker A^m = \ker A^{m+1}$ for some positive integer m, and finite descent if $R(A^n) = R(A^{n+1})$ for some positive integer n. Laursen Proposition 1.8

of []) proved that if $A-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, then A has the single valued extension property, SVEP for short.

We will say in the following that an operator $A \in B(H)$ is conditionally totally posinormal, CTP for short, if to each $\lambda \in \mathbb{C}$ there corresponds an operator $P_\lambda \geq 0$ such that $|(A-\lambda)^*|^2 \leq |P_\lambda^2(A-\lambda I)|^2$; A will be said to be totally posinormal, TP for short, if A is CTP and the positive operator P_λ can be chosen independent of λ.

Lemma 3. *Let $A \in B(H)$ be a CTP operator. Then $A-\lambda I$ has finite ascent for all $\lambda \in \mathbb{C}$. In particular A has the single the single valued extension property.*

Proof. It is easy to see that if $A \in CTP$, then $\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^*$. Hence $ascent(A-\lambda I) \leq 1$ for all $\lambda \in \mathbb{C}$. Thus A has SVEP. □

Since an operator $A \in TP$ has Bishop's property (β), the proof of the following lemma is immediate.

Lemma 4. *Let $A \in TP$. Then A has the single valued extension property.*

Theorem 5. *Let $A \in B(H)$ has SVEP and let $\lambda \in \sigma(A)$ be an isolated point of $\sigma(A)$. Then*

$$X_A(\{\lambda\}) = \{x \in H : ||(A-\lambda)^n x||^{\frac{1}{n}} \to 0\} = E_\lambda H,$$

where E_λ denotes the Riesz idempotent for λ. In particular the above equalities hold for TP or CTP operator.

Proof. Since A has the single valued extension property, the first equality follows from Corollary 2.4 of [] and the second equality follows from ([], p.424). □

Proposition 6. *Let $A \in B(H)$ be a CTP or a TP operator and $\mathcal{M} \subset H$ be an invariant subspace of A. Then the restriction $A|_{\mathcal{M}}$ is also CTP or TP.*

Proof. Let P be the orthogonal projection on $\mathcal{M}$. Then for all $z \in \mathbb{C}$ and for all $x \in H$,

$$||(A-zI|_{\mathcal{M}})^*x|| = ||P(A-z)^*x|| = ||(A-zI)^*x|| = \mathcal{M}_z||(A|_{\mathcal{M}}-zI)x||.$$

□

Lemma 7. [, Lemma 10] *Let $A \in B(H)$ be a TP operator. If $\sigma(A-\lambda I) = 0$, then $A-\lambda I = 0$.*

Lemma 8. *Let A be a quasinipotent algebraically posinormal operator. Then A is nilpotent.*

Proof. Assume that $p(A)$ is totally posinormal for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A)-p(0)$ is quasinilpotent. Thus Lemma 2.3 implies that

$$cA^m(A-\lambda_1)(A-\lambda_2)\cdots(A-\lambda_n) \equiv p(A)-p(0) = 0,$$

where $m \geq 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. □

It is known that SVEP is stable under the functional calculus, i.e. if $A \in B(H)$ has SVEP, then so does $f(A)$ for each function f analytic in an neighborhood of $\sigma(A)$.

Lemma 9. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ has SVEP for each function f analytic in a neighborhood $\sigma(A)$.*

Theorem 10. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ satisfies Browder's theorem for each function f analytic in a neighborhood $\sigma(A)$.*

Proof. It is known that operators with SVEP satisfy Browder's theorem []. Then $f(A)$ satisfies Browder's theorem. This completes the proof. □

We recall the following well known theorems which will be used for the sequel.

Theorem 11. [] *Let $A \in B(H)$ be totally posinormal. Then generalized Weyl's theorem holds for A.*

Theorem 12. *Let $A \in B(H)$ be totally posinormal. Then $f(A)$ satisfies generalized Weyl's theorem for every function f analytic in a neighborhood of $\sigma(A)$. In particular, Weyl's theorem holds for $f(A)$.*

Restricting themselves to only those $A \in CTP$ for which the spectrum $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I) = 0$ for every $M \in Lat(A)$, Jeon *et al* [, Proposition 3.5] have shown that A satisfies Weyl's theorem. In the following theorems we can give more.

Theorem 13. [] *Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I) = 0$ for every $M \in Lat(A)$. Then generalized Weyl's theorem holds for A.*

Remark 14. *Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I) = 0$ for every $M \in Lat(A)$. If A is quasinilpotent, then Lemma 2.4 implies that A is nilpotent. By using the same techniques used in the proof of* [, Lemma 2.3]*, it is easy to see that A is isoloid.*

Theorem 15. [] *Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I) = 0$ for every $M \in Lat(A)$. Then $f(A)$ satisfies the generalized Weyl's theoren for each function f analytic in a neighborhood of $\sigma(A)$. In particular Weyl's theorem holds for $f(A)$.*

The essential approximate point spectrum $\sigma_{ea}(A)$ is defined by

$$\sigma_{ea}(A) = \bigcap\{\sigma_{ap}(A+K) :\ K \text{is a compact operator}\}$$

where $\sigma_{ap}(A)$ is the approximate point spectrum of A. We consider the set

$$\Phi_{+}^{-}(H) = \{A \in B(H) :\ A \text{is left semi-Fredholm and ind} A \leq 0\}.$$

V. Rakocevic [] proved that

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} :\ A - \lambda \notin \Phi_{+}^{-}\}$$

and the inclusion $\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A))$ holds for all function $f(z)$ which are analytic on some open neighborhood of $\sigma(A)$ with no restriction on A. The next theorem shows the spectral mapping theoren on the essential approximate point spectrum of conditionally totally posinormal or totally posinormal operator.

Lemma 16. *Let $A \in B(H)$ and let $\lambda \in \mathbb{C}$. If $A - \lambda$ is semi-Fredholm and it has finite ascent, then ind$(A - \lambda) \leq 0$.*

Proof. If $A - \lambda$ has finite descent, then ind$(A - \lambda) = 0$ by Theorem V 6.2 of []. If $A - \lambda$ does not have finite descent, then

$$n\text{ind}(A - \lambda) = \dim N(A - \lambda)^n - \dim R((A - \lambda)^n))^{\perp} \to -\infty.$$

Hence ind$(A - \lambda) < 0$. □

Corollary 17. *Let $A \in B(H)$ be conditionally totally posinormal operator or totally posinormal. If $A - \lambda I$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then ind$(A - \lambda I) \leq 0$.*

Theorem 18. []
Let $A \in B(H)$ has SVEP. Then

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every function $f(z)$ which is analytic on some open neighborhood G of $\sigma(A)$.

Corollary 19. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every function $f(z)$ which is analytic on some open neighborhood G of $\sigma(A)$.

We say that a-Browder's theorem holds for A if $\sigma_{ea}(A) = \sigma_{ab}(A)$. It is well known that

$$a - \text{Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

In general [] Weyl's theorem does not hold for operators having SVEP(single valued extension property) only but a-Browder's theorem holds for operators having the single valued extension property only as we will show in Theorem 2.4.

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