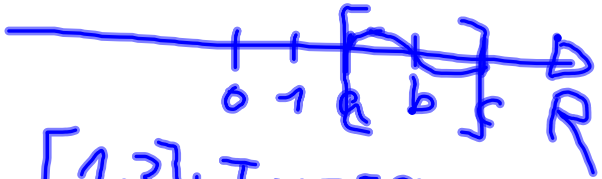
in $\mathbb{R}$

$$\int_0^3 x^2\,dx = \int_0^1 x^2\,dx + \int_1^3 x^2\,dx$$

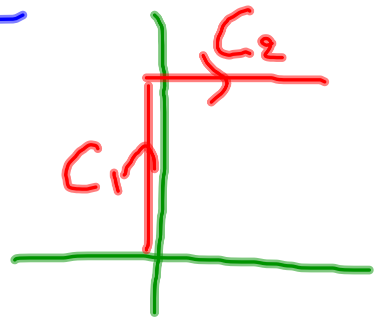
$$\int_a^b + \int_b^c = \int_a^c f(x)\,dx$$



[1,3]: INTERVAL

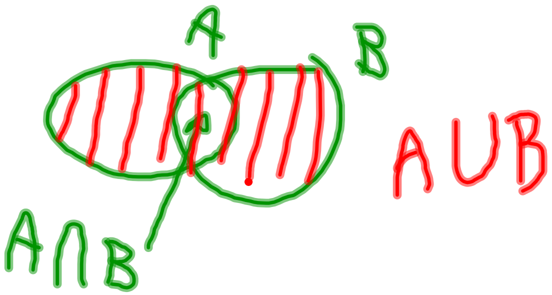
in $\mathbb{C}$

$$C = C_1 \cup C_2$$



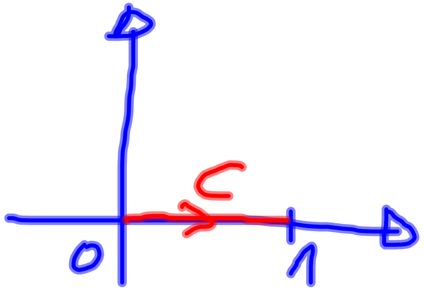
$$\int_C = \int_{C_1} + \int_{C_2} f(z)\,dz$$

C = CONTOUR



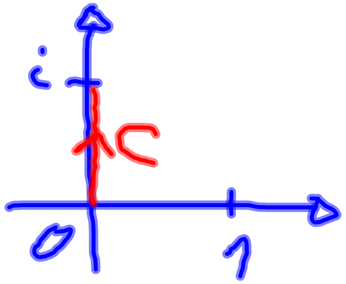
Examples of contours

1-



C: $0 \le x \le 1$

2-

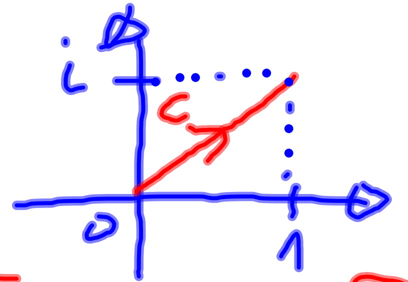


C: $0 \le y \le 1$

$z = x + iy$

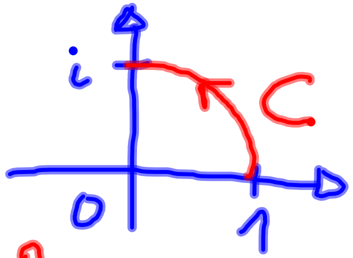
3.

$z = re^{i\theta}$



C: $\theta = \frac{\pi}{4}$; $0 \le r \le \sqrt{2}$

4.

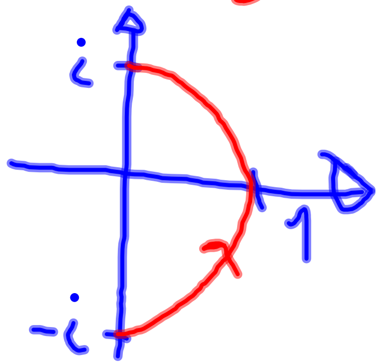


C: $z = e^{i\theta}$; $0 \le \theta \le \frac{\pi}{2}$

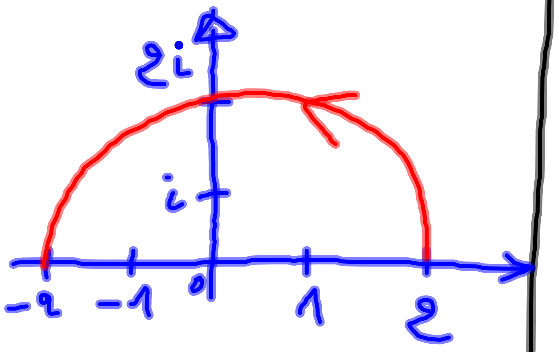
5.

C: $z = e^{i\theta}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



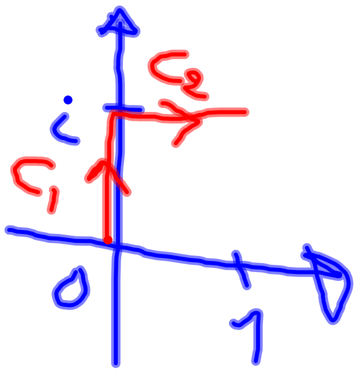
6.



$$C: z = 2e^{i\theta}\,; \quad 0 \le \theta \le \pi$$

Ex: Calculate the integral

$$I = \int_C z^2\,dz$$



$I = I_1 + I_2$ with

$$I_1 = \int_{C_1} z^2\,dz \text{ and } I_2 = \int_{C_2} z^2\,dz$$

$C_1: z = iy$ with $0 \le y \le 1$

$dz = i\,dy$

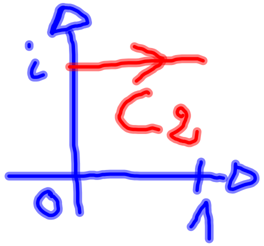
$$I_1 = \int_0^1 (-y^2)\, i\,dy$$

$$= -i \int_0^1 y^2\,dy = -i\left[\frac{y^3}{3}\right]_0^1$$

$$\boxed{I_1 = -\frac{i}{3}}$$

$C_2 : z = x + i$

with $0 \le x \le 1$



$$I_2 = \int_0^1 (x+i)^2 \, dx$$

$$= \left[\frac{(x+i)^3}{3} \right]_0^1$$

$$= \frac{1}{3}\left[(1+i)^3 - i^3 \right]$$

$$I_2 = \frac{1}{3}\left[1 + 3i - 3 \right]$$

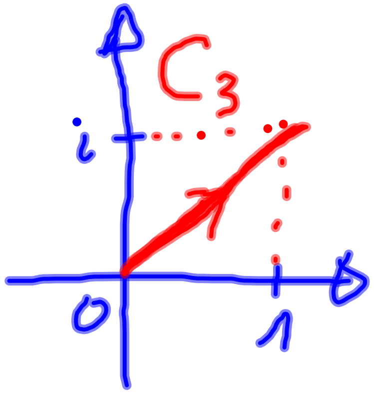
$$= \frac{1}{3}\left[-2 + 3i \right]$$

$$I_2 = i - \frac{2}{3}$$

$$I = I_1 + I_2 = -\frac{i}{3} + i - \frac{2}{3}$$

$$= +\frac{2}{3} i - \frac{2}{3} = \frac{2}{3}(i - 1) = I$$

$$J = \int_{C_3} z^2 \, dz$$



$$C_3 : z = r e^{i\frac{\pi}{4}}$$

$$0 \leq r \leq \sqrt{2}$$

$$dz = e^{i\frac{\pi}{4}} dr$$

$$J = \int_0^{\sqrt{2}} r^2 e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{4}} dr$$

$$= e^{i\frac{3\pi}{4}} \int_0^{\sqrt{2}} r^2 dr$$

$$= \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \left[\frac{r^3}{3}\right]_0^{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{\cancel{2}} (i-1) \frac{\sqrt{2} \times \cancel{2}}{3}$$

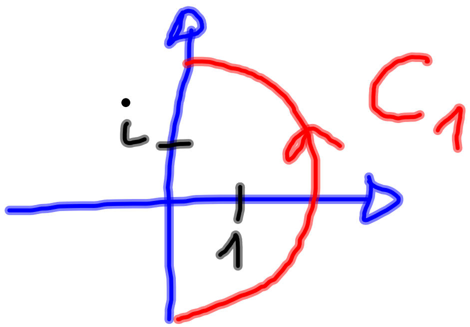
$$\boxed{J = \frac{2}{3}(i-1)}$$

$$\int_0^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$$

$$= \frac{1}{3}\left[1 - i + 3i - 3\right]$$

$$= \frac{1}{3}\left[-2 + 2i\right]$$

$$= \frac{2}{3}\left[i - 1\right]$$



$$I_1 = \int_{C_1} \frac{dz}{z}$$

$$C_1:\ z = 2e^{i\theta};\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$dz = 2ie^{i\theta}\,d\theta$$

So $\dfrac{dz}{z} = i\,d\theta$

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\,d\theta = i\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$
$$= i\pi$$

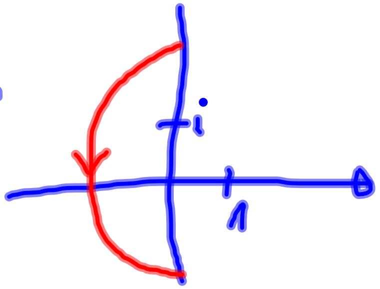
Second Method:

$$I_1 = \Big[\operatorname{Log} z\Big]_{-2i}^{2i} = \operatorname{Log} 2i - \operatorname{Log} -2i$$

$$= \ln 2 + i\frac{\pi}{2} - \left(\ln 2 - i\frac{\pi}{2}\right)$$

$$= i\pi$$

$$I_2 = \int_{C_2} \frac{dz}{z}$$

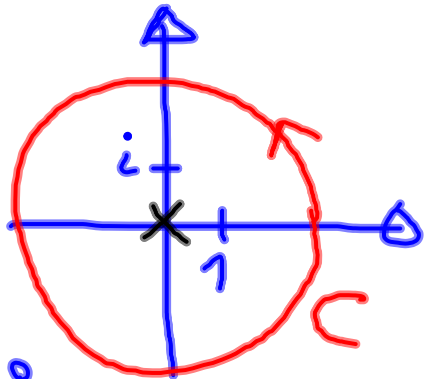


$$I_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} i\, d\theta$$

$$= i\pi$$

Now

$$I = I_1 + I_2 = \oint_C \frac{dz}{z}$$



$$I = I_1 + I_2$$

$$\boxed{\oint_C \frac{dz}{z} = 2i\pi}$$

"Normaly" (if f is analytic inside C)

$$\oint_C f(z)\,dz = 0$$ (Theorem) (Cauchy-Goursat)

if f is analytic in C

but $f(z) = \frac{1}{z}$ is not analytic in $z=0$

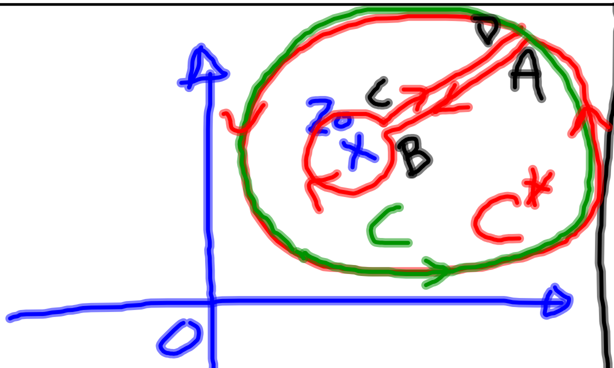
$z=0$ is a pôle, see

$$\oint_C f(z)\,dz = 2i\pi$$

Cauchy integral formula

f analytic inside C (simple closed contour) and z_0 inside C, so:

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2i\pi f(z_0)$$



z_0 inside C but

z_0 not inside C^*.

we have

$$I^* = \int_{C^*} \frac{f(z)}{z - z_0}\, dz = 0$$

(C.GTh)

$$I^* = I_C + \underbrace{I_{AB} + I_{CD}}_{} + I_{\varepsilon^-}$$

$$I_{\varepsilon}^{+} = \int_{\varepsilon} \frac{f(z)}{z - z_0}\, dz$$

$\varepsilon: z = z_0 + \varepsilon e^{i\theta}; \quad 0 \le \theta \le 2\pi$

$$I_{\varepsilon^+} = \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{i\theta})\, dz}{\varepsilon e^{i\theta}}$$

$$z = z_0 + \varepsilon e^{i\theta}$$

So $dz = i\varepsilon e^{i\theta} d\theta$

and

$$I_{\varepsilon+} = \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{i\theta})\, i \cancel{\varepsilon} \cancel{e^{i\theta}}\, d\theta}{\cancel{\varepsilon} \cancel{e^{i\theta}}}$$

$$= i\int_0^{2\pi} f(z_0+\varepsilon e^{i\theta})\, d\theta$$

when $\varepsilon \to 0$: $I_{\varepsilon+} \to 2\pi i f(z_0)$

So when $\varepsilon \to 0$

$\bar{I}_{AB} + \bar{I}_{CD} = 0$ and

$$I^* = I - 2\pi i f(z_0) = 0$$

So

$$\boxed{I = \oint_C \frac{f(z)}{z-z_0} dz = 2i\pi f(z_0)}$$

This is the Cauchy integral Formula ☺

Derivatives of $f(z)$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z}\, ds$$

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(s)\,ds}{(s-z)^2}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(s)\,ds}{(s-z)^{n+1}}$$

Taylor series:

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$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \; ; \; n \in \mathbb{N}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \dots$$

Ex:

$$f(z) = e^z = 1 + z + \frac{1}{2} z^2 + \dots + \frac{1}{n!} z^n + \dots$$

Ex:

$$f(z) = \frac{1}{1-z}$$

$$= 1 + z + z^2 + \dots + z^n + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{iz} = 1 + iz - \frac{z^2}{2} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{-iz} = 1 - iz - \frac{z^2}{2} + \frac{iz^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{iz} - e^{-iz} = 2iz - \frac{2iz^3}{3!} + \dots$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

with the same method:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

What about the func$^{\circ}$

$$f(z) = \frac{e^z}{z} = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$