MATH203 Calculus

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Power series representations of functions

Consider the power series

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^{n-1} + \ldots$$

It is a geometric series with $a = 1$ and $r = x$, if $|x| < 1$, then the sum of series $s = \frac{1}{1-x}$, i.e. $1 + x + x^2 + \cdots + x^{n-1} + \cdots = \frac{1}{1-x}$ if $-1 < x < 1$

Remarks:

1. We can say that the function $f(x) = \frac{1}{1-x}$ is defined by the power series $\sum_{n=1}^{\infty} x^{n-1}$.
2. We can say that $\sum_{n=1}^{\infty} x^{n-1}$ is a power series representation for $f(x) = \frac{1}{1-x}$ if $|x| < 1$.
3. We can say $f(x) = \frac{1}{1-x}$ is representation of function as power series $\sum_{n=1}^{\infty} x^{n-1}$.

Theorem

Suppose that a power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$ then the function defined by

$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$ is differentiable continuous on $(-R, R)$ and

1. $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \ldots$. In other words, the series can be differentiated term by term.
2. $\int_0^x f(x) = C + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \cdots + a_n \frac{x^{n+1}}{n+1} + \ldots$. In other words, the series can be integrated term by term.
3. Note that whether we differentiate or integrate, the radius of convergence is preserved. However, convergence at the endpoints must be investigated every time.

Examples

Find a function representation f of the following power series:

(1): $\sum_{n=0}^{\infty}(-1)^n x^n$

(2): $\sum_{n=0}^{\infty} x^n$

(3): $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

Solution:

Examples

Find a power series representation for the following functions:
(1): $f(x) = \frac{1}{1-x^2}$
(2): $f(x) = \frac{1}{1-4x^2}$
(3): $f(x) = \frac{2}{(1+2x)^2}$
Solution:

Note: All the power series considered in previous lecture are called power series centered at 0.

Examples

Find a power series representation for the following functions:

(1): $f(x) = \ln(1 + x)$

(2): $f(x) = e^x$

(3): $f(x) = \frac{x^3}{4-x^3}$

(4): $f(x) = \frac{5}{3-x}$ centered at (a) $c = 1$, (b) $c = 0$

(5): $f(x) = \frac{(x-3)^2}{4-2x}$ centered at $c = 3$

Solution: