MATH203 Calculus

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Theorem 2

If a series $\sum_{n=1}^{\infty} a_n$ is c'gt, then $\lim_{n\to\infty} a_n = 0$.

Theorem 3 (nth-term test)

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is d'gt.

Theorem 4

If two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are such that $a_i = b_i$ for every $i > k$, where k is a positive interger, then both series converge or diverge together.

Theorem 5

If we delete first k terms of a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_k + \cdots + a_n + \ldots$ then its behaviour does not change.

Theorem 6 (properties)

Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ and C is a real number, then

- $\sum_{n=1}^{\infty} C a_n = C \sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$.

Theorem 7

If $\sum_{n=1}^{\infty} a_n$ is convergent, and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty}(a_n + b_n)$ is divergent.

Examples

In page (26) (i): $3 + \frac{3}{4} + \cdots + \frac{3}{(4)^{n-1}} + \ldots$

(ii): $\sum_{n=1}^{\infty}(\sqrt{2})^{n-1}$

Solution:

Examples

In page (27) Q25: $\sum_{n=1}^{\infty} a_n = \frac{1}{4*5} + \frac{1}{5*6} + \cdots + \frac{1}{(n+3)(n+4)} + \dots$

Q28: $\sum_{n=1}^{\infty} a_n = \frac{-1}{1*2} + \frac{-1}{2*3} + \cdots + \frac{-1}{n(n+1)} + \dots$

Solution:

Examples

In page (28) Q1: $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{(5n-1)}$

Q2: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{e}}$

Q4: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\ln(n+1)}$

Solution:

Def of Positive Term Series

a series $\sum_{n=1}^{\infty} a_n$ such that $a_n > 0$ for every n

Theorem 1

If $\sum_{n=1}^{\infty} a_n$ is a positive term series and if there exists a number M such that $S_n = a_1 + a_2 + \cdots + a_n < M$ for every n, then the series is c'gt and has sum $S \leqslant M$. If no such M exists, then the series is d'gt.

Theorem 2 (Integral Test)

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series. Suppose also

- f is a positive continuous function for $x \geqslant 1$ such that
- $f(n) = a_n$, for $n = 1, 2, 3, \ldots$
- f is a decreasing function of interval $[1, \infty)$

then, $\sum_{n=1}^{\infty} a_n$ is c'gt if $\int_1^{\infty} f(x)dx$ is c'gt

and $\sum_{n=1}^{\infty} a_n$ is d'gt if $\int_1^{\infty} f(x)dx$ is d'gt

Theorem 3 ($p-$Series Test)

The $p-$series is given by $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots$, where $p > 0$ by definition.

- If $p > 1$, then the series converges.
- If $0 < p \leqslant 1$, then the series diverges.

Theorem 4 (Basic Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two positive term series. If $0 \leqslant a_n \leqslant b_n$ for all n, then the following rules apply:

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ an converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ an diverges.

Theorem 5 (Limit Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two positive term series. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$, then both series converge or diverge together.

Examples

In page (34) Q2: $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

Q3: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Q4: $\sum_{n=1}^{\infty} \frac{arc\tan n}{1+n^2}$

Solution:

Examples

In page (38) (i): $\sum_{n=1}^{\infty} \frac{1}{5+6^n}$ (hint: using direct CT)

(ii): $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}+1}$ (hint: using direct CT)

In page (39) (i): $\sum_{n=1}^{\infty} \frac{1}{1+e^{2n}}$ (hint: using Limit CT)

(ii): $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{6+n^2+n^{7/2}}$ (hint: using Limit CT)

Solution:

The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then

- If $L < 1$ the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$ the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$ (fails), the series may converge or diverge.

Examples:

(1) $\sum_{n=1}^{\infty} n!$ (2) $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ (3) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ (4) $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$.

The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, then

- the series $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$.
- the series $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$.
- If $L = 1$ (fails), the series may converge or diverge.

Examples:

(1) $\sum_{n=1}^{\infty} \frac{5^n}{n^n}$ (2) $\sum_{n=1}^{\infty} \left(\frac{8n^2-7}{n+1}\right)^n$ (3) $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$