MATH203 Calculus

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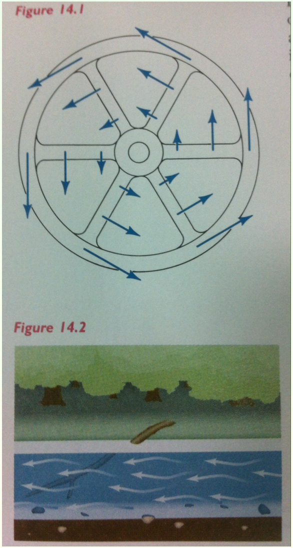
Vector Fields

- If to each point K in a region there is assigned exactly one vector having initial point K, then the collection of all such vectors is a **vector field**.
- A vector field in which each vector represents velocity is a called a **velocity field**.
- A vector field in which each vector represents force is a called a **force field**, i.e. mechanics and electricity.
- **Steady vector fields** is a vector field in which every vector is independent of time,

Vector Fields



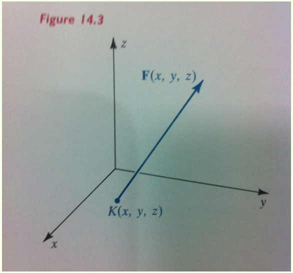
Vector Fields

Definition: Vector field in $3-$dimensions

A vector field in three dimensions is a function $\mathbf{F}$ whose domain D is a subset of $\mathbb{R}^3$ and whose range is a subset of V_3. If $(x, y, z) \in D$, then

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k},$$

where M, N and P are scalar functions.

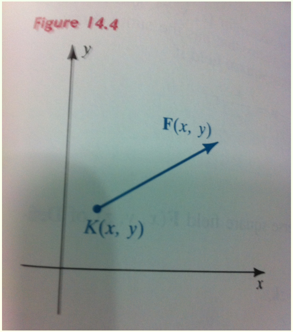


Vector Fields

Definition: Vector field in $2-$dimensions

A vector field in two dimensions is a function $\mathbf{F}$ whose domain D is a subset of $\mathbb{R}^2$ and whose range is a subset of V_2. If $(x, y) \in D$, then

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j},$$



Vector Fields

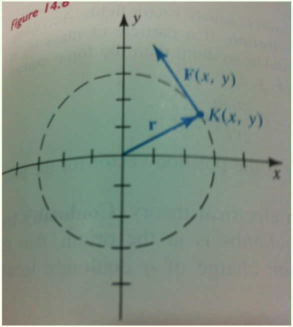
Example: Describe the vector field $\mathbf{F}$ if $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

Notes:

1- $\mathbf{F}(x, y)$ is tangent to the circle of radius r with center at origin.

2- $\|\mathbf{F}(x, y)\| = \sqrt{x^2 + y^2} = \|r\|$.

3- This implies that $\|\mathbf{F}(x, y)\|$ increases as the point $P(x, y)$ moves away from the center.



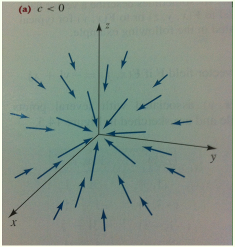
Vector Fields

Definition: Inverse Square Field

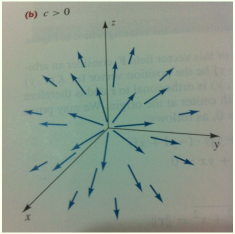
Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector for (x, y, z) and let the vector $\mathbf{u} = \frac{1}{\|\mathbf{r}\|}\mathbf{r}$ has the same direction as $\mathbf{r}$. A vector field $\mathbf{F}$ is an **inverse square field** if

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^2}\mathbf{u} = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r},$$

where c is a scalar.



(a) $c < 0$



(b) $c > 0$

Vector Fields

Definition: Gradient of a function

If f is a function of three variables, the gradient of $f(x,y,z)$ is the following vector field

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

Definition: Conservative

A vector field $\mathbf{F}$ is conservative, if

$$\mathbf{F}(x,y,z) = \nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

for some scalar function f.

Definition: Potential function

If A vector field $\mathbf{F}$ is conservative, then f is called a **Potential function** for $\mathbf{F}$ and $f(x,y,z)$ is potential at the point (x,y,z).

Vector Fields

Theorem

Every inverse sqaure vector field is conservative.

Vector differential in $3-$dimensions

$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

Notes:

(1) ∇ operating on a scalar function f, produces then gradient of f, i.e.

$$\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

(2) ∇ operating on a vector field to define another vector field called the **curl** of $\mathbf{F}$, denoted by

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

Vector Fields

Definition: curl

Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$, where M, N and P have partial derivatives in some region. Then,

$$\textbf{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Definition: divergence

Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$, where M, N and P have partial derivatives in some region. Then,

$$\textbf{div}(\mathbf{F}) = \nabla.\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Vector Fields

Test for conservative vector field in space

Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is a vector field in space, where M, N and P have continuous first partial derivatives in an open region. Then, $\mathbf{F}$ is conservative if and only if

$$\text{curl}(\mathbf{F}) = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Vector Fields

Examples

(1) If $\mathbf{F}(x,y,z) = xy^2z^4\mathbf{i} + (2x^2y+z)\mathbf{j} + y^3z^2\mathbf{k}$, Find $\nabla \times \mathbf{F}$ and $\nabla.\mathbf{F}$

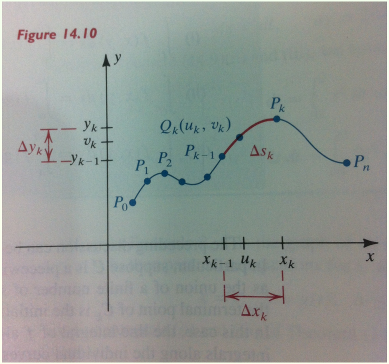
(2) Find a potential function for a conservative vector field
(a) $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$,
(b) $\mathbf{F}(x,y,z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$
(c) $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2zy\mathbf{k}$

(3) Find conservative vector field that has given potential
(a) $f(x,y,z) = x^2 - 3y^2 + 4z^2$
(b) $f(x,y,z) = \sin(x^2 + y^2 + z^2)$

(4) Find $\nabla \times \mathbf{F}$ and $\nabla.\mathbf{F}$
(a) $\mathbf{F}(x,y,z) = x^2z\mathbf{i} + y^2x\mathbf{j} + (y+2z)\mathbf{k}$,
(b) $\mathbf{F}(x,y,z) = 3xyz^2\mathbf{i} - y^2\sin z\mathbf{j} - xe^{2z}\mathbf{k}$

Line Integrals

We shall study the new type of integral called a **Line integral** defined by $\int_c f(x,y)ds$. Recall that a plane curve C is smooth if it has parametrisation $x = g(t)$, $y = h(t)$; $a \leqslant t \leqslant b$



Line Integrals

Let $\Delta x_k = x_k - x_{k-1}$, $\Delta y_k = y_k - y_{k-1}$ and $\Delta s_k = \text{length of } \widehat{P_{k-1}P_k}$.

Definition: **Line Integrals in Two Dimensions**

$$\int_C f(x,y)ds = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k)\Delta s_k$$

$$\int_C f(x,y)dx = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k)\Delta x_k$$

$$\int_C f(x,y)dy = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k)\Delta y_k$$

where $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{[g'(t)]^2 + [h'(t)]^2}dt$,
$dx = g'(t)dt$ and $dy = h'(t)dt$

Line Integrals

Evaluation theorem for Line integral

If a smooth curve C is given by $x = g(t)$, $y = h(t)$; $a \leqslant t \leqslant b$ and $f(x, y)$ is continuous on region D containing C, then

$$\text{(i)} \int_C f(x,y)ds = \int_a^b f(g(t),h(t))\sqrt{[g'(t)]^2 + [h'(t)]^2}dt$$

$$\text{(ii)} \int_C f(x,y)dx = \int_a^b f(g(t),h(t))g'(t)dt$$

$$\text{(iii)} \int_C f(x,y)dy = \int_a^b f(g(t),h(t))h'(t)dt$$

Line Integrals

Mass of a wire

$$m = \int_C \delta(x, y) ds$$

Examples

(1) Evaluate $\int_C xy^2 ds$ if C has parametrisation $x = \cos t, y = \sin t; 0 \leqslant t \leqslant \pi/2$.

(2) Evaluate $\int_C xy^2 dx$ and $\int_C xy^2 dy$ if C has parametrisation $y = x^2$ from $(0, 0)$ to $(2, 4)$.

(3) Evaluate $\int_C (x^2 - y + 3z) ds$ if C has parametrisation $x = t, y = 2t$ and $z = t$ from $0 \leqslant t \leqslant 1$.

(4) A thin wire is bent into the shape of a semicircle of radius a with descity $\delta = ky$. Find the mass of the wire.

Line Integrals

Examples

(1) Evaluate $\int_C xydx + x^2dy$ if

(a) C consist of line segment from (2,1) to (4,1) and (4,1) to (4,5).

(b) C is the line segment from (2,1) to (4,5).

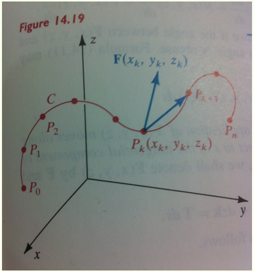
(c) C has parametrisation $x = 3t - 1, y = 3t^2 - 2t; 1 \leqslant t \leqslant 5/3$.

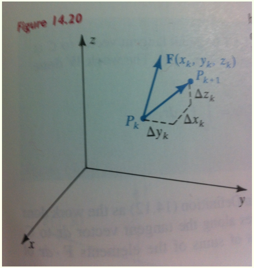
Line Integrals

Definition: Work done

$$W = \lim_{\|P\|\to 0} \sum_k \Delta W$$
$$= \int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$

This is the line integral represents the work done by the force **F** along to the curve C.





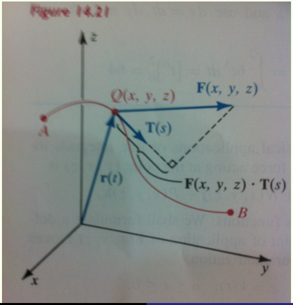
Line Integrals

Definition

Let C be a smooth space curve, Let $\mathbf{T}$ be a unit tangential vector to C at (x, y, z), and let **F** be force acting at (x, y, z). The **work done by F along C** is

$$W = \int_C \mathbf{F}.\mathbf{T}ds = \int_C \mathbf{F}.d\mathbf{r}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.



Line Integrals

Examples

(1) If an inverse force field $\mathbf{F}$ is given by

$$\mathbf{F}(x, y, z) = \frac{k}{\|r\|^3}\mathbf{r},$$

where k is a constant, find the work done by **F** along $x-$axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

(2) Let C be the part of the parapola $y = x^2$ between (0,0) and (3,9). If $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is a force field acting at (x, y), find the work done by **F** along C from

(a) (0,0) to (3,9)

(b) (3,9) to (0,0).

Line Integrals

Examples

(3) The force at a point (x, y) in a coordinate plane is given by $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + xy\mathbf{j}$. Find the work done by **F** along the graph of $y = x^3$ from $(0, 0)$ to $B(2, 8)$.

(4) The force at a point (x, y, z) in three dimensions is given by $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. Find the work done by **F** along the twisted cubic $x = t$, $y = t^2$ $z = t^3$ from $(0, 0, 0)$ to $B(2, 4, 8)$.

(5) Evaluate $\int_C xyzds$ if C is the line segment from (0,0,0) to (1,2,3).