THE VIRIAL THEOREM

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Given a system having α particles, with associated position $\vec{r}_\alpha$ and momenta $\vec{p}_\alpha$. We define the virial function as:

$$\varsigma = \sum_\alpha \vec{p}_\alpha \cdot \vec{r}_\alpha \tag{1}$$

It would be interesting to look for the time derivative of this function:

$$\frac{d\varsigma}{dt} = \sum_\alpha \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha + \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \tag{2}$$

Since we are dealing with many-particle system. We can take the time average for the previous expression

$$\begin{aligned}\langle \frac{d\varsigma}{dt} \rangle =& \frac{\int_0^\tau \frac{d\varsigma}{dt} dt}{\int_0^\tau tdt} \\ =& \frac{\varsigma(\tau) - \varsigma(0)}{\tau}\end{aligned} \tag{3}$$

Now, if the system has a periodic motion of a period τ. The time average for the derivative of the virial function will vanish. even if the system doe not admit a periodic motion, the virial function ought to be bounded, hence one can integrate over a sufficiently large interval such that the time average $\langle \frac{d\varsigma}{dt} \rangle$ will approach zero. Hence, we have (at least as an approximation):

recall that $\vec{F} = \dot{\vec{p}}$

$$\langle \sum_\alpha \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \rangle = -\langle \sum_\alpha \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha \rangle \tag{4}$$

We can now identify the LHS being twice the kinetic energy , the RHS is the force dotted with the position :

$$\langle T \rangle = -\frac{1}{2} \langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \rangle \tag{5}$$

This is the **Virial theorem** , the expected value for the kinetic energy for a system is equal to its virial function.
It is interesting to look at forces that arise from central potential taking the form :

$$V = kr^{n+1} \tag{6}$$

Hence, by the virial theorem eq (5):

$$\begin{aligned}\langle T \rangle =& \frac{1}{2} \langle r \cdot \frac{d}{dr}(kr^{n+1}) \rangle \\ =& \frac{1}{2} \langle (n+1) kr^{n+1} \rangle \\ =& \frac{n+1}{2} \langle V \rangle\end{aligned} \tag{7}$$

For Columb and gravitational potentials, $n = -2$. Therefore we have :

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle \tag{8}$$