Q1) The time to failure for a computer printer fan has a Weibull distribution with shape parameter α = 2 and scale parameter β= 3. Testing has indicated that the distribution is limited to the range from 1.5 to 4.5. Generate 1 failure time for this this truncated distribution at U= 0.943.

Solution:
Notice that the range is truncated, we have:

F(1.5) = 1-exp(-(1.5/3)^2) = 0.22119

F(4.5) = 1-exp(-(4.5/3)^2) = 0.8946

W = 0.22119 + (0.8946 – 0.22119)*0.943 = 0.8562169

X = 3[-ln(1-0.8562169)]^(1/2) = 4.1779

Q2) Customers arrive at a service location according to a Poisson distribution with mean 10 per hour. The installation has two servers. Experience shows that 60% of the arriving customers prefer the first server. By using random numbers (0,1) given

0.943	0.398	0.372	0.943	0.204	0.794
0.498	0.528	0.272	0.899	0.294	0.156
0.102	0.057	0.409	0.398	0.400	0.997

Determine the arrival times of the first three customers at each server.

Solution:

	A	B	C	D	E	F
1	customer	U	Inter-Arrival Time	Arrival time	U	server
2	1	0.943	0.2864704	0.2864704	0.498	1
3	2	0.102	0.01075852	0.29722892	0.398	1
4	3	0.528	0.07507763	0.37230655	0.057	1
5	4	0.372	0.04652151	0.41882806	0.272	1
6	5	0.409	0.05259393	0.47142199	0.943	2
7	6	0.899	0.22926348	0.70068547	0.398	1
8	7	0.204	0.02281561	0.72350107	0.294	1
9	8	0.4	0.05108256	0.77458364	0.794	2
10	9	0.156	0.01696028	0.79154392	0.997	2

	A	B	C	D	E	F
1	customer	U	Inter-Arrival Time	Arrival time	U	server
2	1	0.943	=-(1/10)*LN(1-B2)	=C2	0.498	=IF(E2<=0.6,1,2)
3	2	0.102	=-(1/10)*LN(1-B3)	=D2+C3	0.398	=IF(E3<=0.6,1,2)
4	3	0.528	=-(1/10)*LN(1-B4)	=D3+C4	0.057	=IF(E4<=0.6,1,2)
5	4	0.372	=-(1/10)*LN(1-B5)	=D4+C5	0.272	=IF(E5<=0.6,1,2)
6	5	0.409	=-(1/10)*LN(1-B6)	=D5+C6	0.943	=IF(E6<=0.6,1,2)
7	6	0.899	=-(1/10)*LN(1-B7)	=D6+C7	0.398	=IF(E7<=0.6,1,2)
8	7	0.204	=-(1/10)*LN(1-B8)	=D7+C8	0.294	=IF(E8<=0.6,1,2)
9	8	0.4	=-(1/10)*LN(1-B9)	=D8+C9	0.794	=IF(E9<=0.6,1,2)
10	9	0.156	=-(1/10)*LN(1-B10)	=D9+C10	0.997	=IF(E10<=0.6,1,2)
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Q3) Consider the triangular distribution:

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & a \le x \le c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & c < x \le b \\ 1 & b < x \end{cases}$$

a) Derive an inverse transform algorithm for this distribution.
b) Using random numbers U1 = 0.943 and U2 = 0.398 to generate 2 random numbers from the triangular distribution with a = 2, c = 5, b = 10.

Solution:

a)

Proof The triangular(a, c, b) distribution has probability density function

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & a < x < c \\ \frac{2(b-x)}{(b-a)(b-c)} & c \le x < b \end{cases}$$

and cumulative distribution function

$$F(x) = \begin{cases} \frac{(x-a)^2}{(b-a)(c-a)} & a < x < c \\ 1 - \frac{(x-b)^2}{(b-a)(b-c)} & c \le x < b. \end{cases}$$

Equating the cumulative distribution function to u, where $0 < u < 1$ yields an inverse cumulative distribution function

$$F^{-1}(u) = \begin{cases} a + \sqrt{(b-a)(c-a)u} & 0 < u < \frac{c-a}{b-a} \\ b - \sqrt{(b-a)(b-c)(1-u)} & \frac{c-a}{b-a} \le u < 1. \end{cases}$$

b)

(c-a)/(b-a) = 0.375

For U1 = 0.943, since 0.943 > 0.375, we have X = b-SQRT((b-a)(b-c)*(1-U1)) = 8.8304

For U2 = 0.398, since 0.398 > 0.375, we have X = b-SQRT((b-a)(b-c)*(1-U2)) = 6.1989

Q4) Suppose that the service time for a patient consists of two distributions. There is a 25% chance that the service time is uniformly distributed with minimum of 20 minutes and a maximum of 25 minutes, and a 75% chance that the time is distributed according to a Weibull distribution with shape of 2 and a scale of 4.5. Using random numbers U1 = 0.943, U2 = 0.398, U3 = 0.372 and U4 = 0.943 to generate the service time for two patients.

Solution:

This is a mixture distribution. Let F_1 represent the U(20,25) distribution with $\omega_1 = 0.25$. Let F_2 represent the Weibull distribution with $\omega_2 = 0.75$.

Using U1 = 0.943 to pick the distribution implies, X ~ Weibull because 0.943 > 0.25

$$X = \beta[-ln(1-u)]^{\frac{1}{\alpha}}$$

Using U2 = 0.398

X = 4.5[-ln(1-0.398)]^(1/2) = 3.2057

Using U3 = 0.372 to pick the distribution implies, X ~ Weibull because 0.372 > 0.25

Using U4 = 0.943

X = 4.5[-ln(1-0.943)]^(1/2) = 7.616

	A	B
1	alpha	2
2	beta	4.5
3	u =	=RAND()
4	Finv(U) =	=$B$2*(-1*LN(1-B3))^(1/$B$1)
5		
6	u =	=RAND()
7	a =	20
8	b =	25
9	U(10,20) =	=$B$7+($B$8-$B$7)*B6
10		
11	u =	=RAND()
12	p =	0.25
13	X =	=IF(B11<$B$12,B9,B4)
14		

Q5) Consider the following probability density function:

$$f(x) = \begin{cases} \frac{3x^2}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Derive an acceptance-rejection algorithm for this distribution.
b) Using the first row of random numbers in the top generate 2 random numbers using your algorithm.

Solution:

Choose g(x) = 3/2. Integrating over [-1, 1] yields c = 3. Thus, w(x) = ½ over [-1,1]

Algorithm
Repeat
  Generate W ~ w(x) which is U(-1,1)
  Generate U ~ U(0,1)
Until U*g(W) <= f(W)
Return W

W = a + (b-a)* U = -1 + (1 - -1)U = 2*U -1

U1 = 0.943
W = 2*0.943 -1 = 0.886

U2 = 0.398
Is 0.398*1.5 <= 1.5(0.886)^2?

0.597 < 1.177, therefore accept X = W = 0.886

U1 = 0.372
W = 2*0.372 -1 = -0.256

U2 = 0.943
Is 0.943*1.5 <= 1.5(-0.256)^2?

1.4145 < 0.098304, therefore reject W

Continue in this manner until you get the 2^{nd} acceptance.

	A	B	C
1			
2	W	U	
3	-0.9114397	0.15318094	accept
4	-0.9636792	0.90809909	accept
5	-0.9400854	0.54495939	accept
6	0.1000509	0.45825813	reject
7	-0.4554171	0.62509339	reject
8	-0.4055985	0.30772812	reject
9	-0.3104027	0.04776647	accept
10	0.2478788	0.29668948	reject
11	-0.1916972	0.0813116	reject
12	0.35361853	0.44314624	reject
13	-0.4533482	0.4526399	reject
14	0.15509248	0.61590692	reject
15	-0.1365031	0.60220435	reject
16	-0.77923	0.65644378	reject

	A	B	C
1			
2	W	U	
3	=-1 + (1--1)*RAND()	=RAND()	=IF(B3*1.5<=1.5*(A3)^2,"accept", "reject")
4	=-1 + (1--1)*RAND()	=RAND()	=IF(B4*1.5<=1.5*(A4)^2,"accept", "reject")
5	=-1 + (1--1)*RAND()	=RAND()	=IF(B5*1.5<=1.5*(A5)^2,"accept", "reject")
6	=-1 + (1--1)*RAND()	=RAND()	=IF(B6*1.5<=1.5*(A6)^2,"accept", "reject")
7	=-1 + (1--1)*RAND()	=RAND()	=IF(B7*1.5<=1.5*(A7)^2,"accept", "reject")
8	=-1 + (1--1)*RAND()	=RAND()	=IF(B8*1.5<=1.5*(A8)^2,"accept", "reject")
9	=-1 + (1--1)*RAND()	=RAND()	=IF(B9*1.5<=1.5*(A9)^2,"accept", "reject")
10	=-1 + (1--1)*RAND()	=RAND()	=IF(B10*1.5<=1.5*(A10)^2,"accept", "reject")
11	=-1 + (1--1)*RAND()	=RAND()	=IF(B11*1.5<=1.5*(A11)^2,"accept", "reject")
12	=-1 + (1--1)*RAND()	=RAND()	=IF(B12*1.5<=1.5*(A12)^2,"accept", "reject")
13	=-1 + (1--1)*RAND()	=RAND()	=IF(B13*1.5<=1.5*(A13)^2,"accept", "reject")
14	=-1 + (1--1)*RAND()	=RAND()	=IF(B14*1.5<=1.5*(A14)^2,"accept", "reject")
15	=-1 + (1--1)*RAND()	=RAND()	=IF(B15*1.5<=1.5*(A15)^2,"accept", "reject")
16	=-1 + (1--1)*RAND()	=RAND()	=IF(B16*1.5<=1.5*(A16)^2,"accept", "reject")