Exercise 1

Given a sequence $x(n)$ for $0 \leq n \leq 3$, where $x(0) = 1, x(1) = 1, x(2) = -1$, and $x(3) = 0$, compute its DFT $X(k)$.

Solution 1

We have the DFT coefficients formula: $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad \text{for } k = 0, 1, \cdots, N-1$

Thus, for k=0: $X(0) = \sum_{n=0}^{3} x(n)e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0} = 1 + 1 - 1 + 0 = 1$

For k=1: $X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi}{2}n} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\frac{2\pi}{2}} + x(3)e^{-j\frac{3\pi}{2}} = 1 + 1(-j) - 1(-1) + 0 = 2 - j$

For k=2: $X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} = 1 + 1(-1) - 1(1) + 0 = -1$

For k=3: $X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi}{2}3n} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} = 1 + 1(j) - 1(-1) + 0 = 2 + j$

$$X(0) = 1,\ X(1) = 2 - j,\ X(2) = -1,\ X(3) = 2 + j$$

Exercise 2

Given the DFT sequence $X(k)$ for $0 \le k \le 3$ obtained in Exercise 1, evaluate its inverse DFT $x(n)$.

Solution 2

Since N=4, we have $W_N^{-nk} = e^{j\frac{\pi}{2}nk}$, using IDFT Equation $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi kn}{2}}$

Thus, for n=0: $x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi}{2}k0} = X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}$

$$= \frac{1}{4}[(1)(1) + (2-j)(1) + (-1)(1) + (2+j)(1)] = 1$$

For n=1: $x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi}{2}k} = X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}$

$$= \frac{1}{4}[(1)(1) + (2-j)(j) + (-1)(-1) + (2+j)(-j)] = \frac{1}{4}[1 + 2j + 1 + 1 - 2j + 1] = 1$$

For n=2: $x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k} = X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}$

$$= \frac{1}{4}[(1)(1) + (2-j)(-1) + (-1)(1) + (2+j)(-1)] = \frac{1}{4}[1 - 2 + j - 1 - 2 - j] = -1$$

For n=3: $x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi}{2}3k} = X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}$

$$= \frac{1}{4}[(1)(1) + (2-j)(-j) + (-1)(-1) + (2+j)(j)] = \frac{1}{4}[1 - 2j + 1 + 1 + 2j - 1] = 0$$

Finally, we obtain: $x(0) = 1, \quad x(1) = 1, \quad x(2) = -1, \quad x(3) = 0.$

Exercise 3

Consider a digital sequence sampled at the rate of 20,000 Hz. If we use the 8,000-point DFT to compute the spectrum, determine

a. the frequency resolution;

b. the folding frequency in the spectrum.

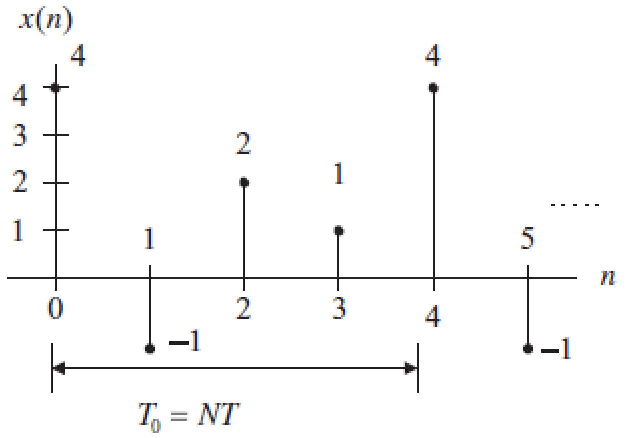
Solution 3

a. We have: $\triangle f = \frac{f_s}{N} = \frac{20000}{8000} = 2.5\ Hz$

b. The folding frequency: $f_{max} = \frac{f_s}{2} = \frac{20000}{2} = 10kHz$

Exercise 4

Given the sequence in Figure and assuming $f_s = 100$ Hz, compute the amplitude spectrum, phase spectrum, and power spectrum.



Solution 4

We have N=4, and $x(0) = 4,\ x(1) = -1,\ x(2) = 2,\ x(3) = 1.$

First we determine the DFT coefficients: $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad \text{for } k = 0, 1, \cdots, N-1$

For k=0: $X(0) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi}{2}n0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0} = (4)(1) + (-1)(1) + (2)(1) + (1)(1) = 6$

For k=1: $X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi}{2}n} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\frac{2\pi}{2}} + x(3)e^{-j\frac{3\pi}{2}} = (4)(1) + (-1)(-j) + (2)(-1) + (1)(j) = 2 + 2j$

For k=2: $X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} = (4)(1) + (-1)(-1) + (2)(1) + (1)(-1) = 6$

For k=3: $X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi}{2}3n} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} = (4)(1) + (-1)(j) + (2)(-1) + (1)(-j) = 2 - 2j$

$$X(0) = 6,\ X(1) = 2 + 2j,\ X(2) = 6,\ X(3) = 2 - 2j$$

Amplitude Spectrum

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(\mathrm{Real}[X(k)])^2 + (\mathrm{Imag}[X(k)])^2}, \quad k = 0, 1, 2, \cdots, N-1$$

$$A_0 = \frac{1}{4}\sqrt{(6)^2+(0)^2} = 1.5, \quad A_1 = \frac{1}{4}\sqrt{(2)^2+(2)^2} = 0.707, \quad A_2 = \frac{1}{4}\sqrt{(6)^2+(0)^2} = 1.5, \quad A_3 = \frac{1}{4}\sqrt{(2)^2+(-2)^2} = 0.707$$

Power Spectrum $$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\left\{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2\right\}, \qquad k = 0, 1, 2, \cdots, N-1$$

Phase Spectrum $$\phi_k = \tan^{-1}\left(\frac{\text{Imag}[X(k)]}{\text{Real}[X(k)]}\right), \quad k = 0, 1, 2, \cdots, N-1$$

f Hz	0	25	50	75
A_k	1.5	0.707	1.5	0.707
P_k	2.25	0.5	2.25	0.5
φ_k degree	0	45	0	-45

Exercise 5

Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 4, x(1) = 3, x(2) = 2$, and $x(3) = 1$, evaluate its DFT $X(k)$ using the decimation-in-frequency FFT method, and determine the number of complex multiplications.

Solution 5

DFT coefficient using decimation-in-frequency FFT method

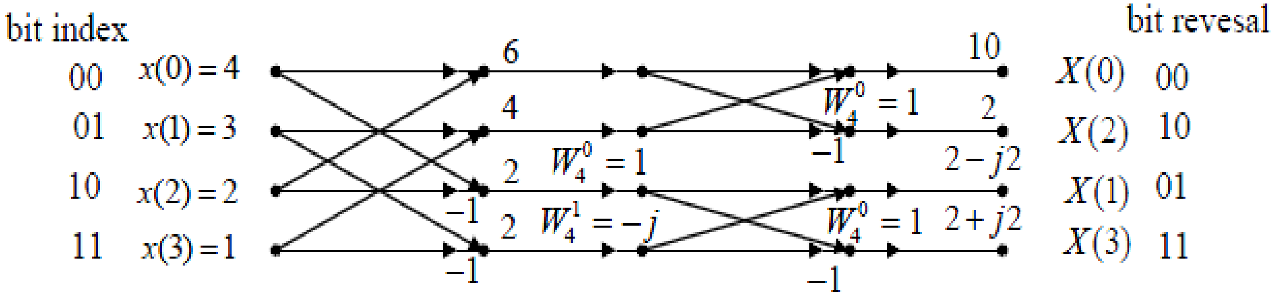
$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n) with (N/2) \text{points}\}$$

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^n W_{N/2}^{mn} = DFT\{b(n) W_N^n \; with \; (N/2) \text{points}\}$$

$$a(n) = x(n) + x\left(n + \frac{N}{2}\right), \quad \text{for } n = 0, 1 \cdots, \frac{N}{2} - 1$$

$$b(n) = x(n) - x\left(n + \frac{N}{2}\right), \quad \text{for } n = 0, 1, \cdots, \frac{N}{2} - 1$$

We have:



Complex multiplications of DFT $= N^2$, and

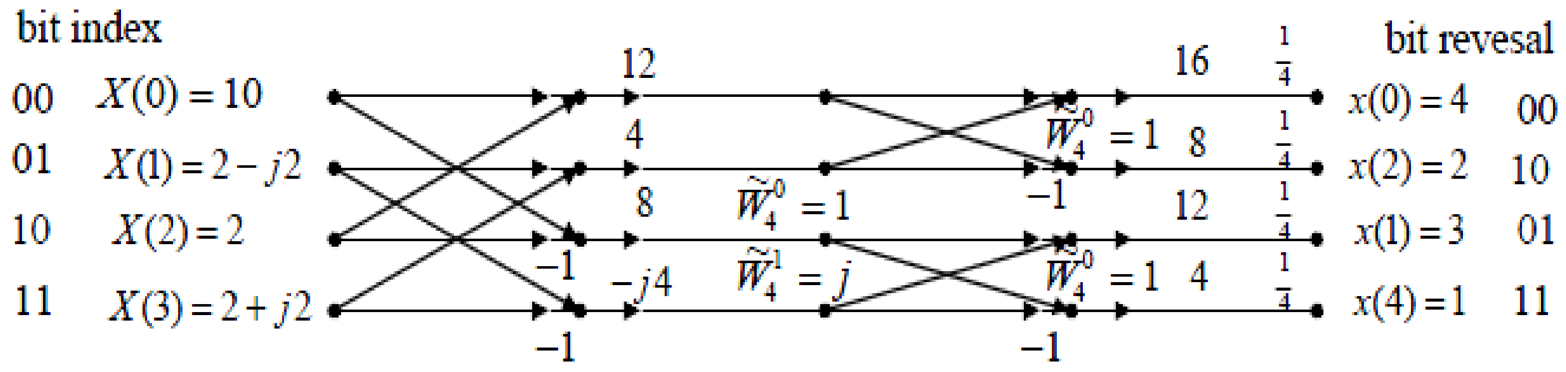
Complex multiplications of FFT $= \dfrac{N}{2}\log_2(N)$

$X(0) = 10$, $X(1) = 2 - 2j$, $X(2) = 2$, $X(3) = 2 + 2j$, 4 complex multiplications

Exercise 6

Given the DFT sequence $X(k)$ for $0 \le k \le 3$ obtained in exercise 5, evaluate its inverse DFT $x(n)$ using the decimation-in-frequency FFT method.

Solution 6



Exercise 7

Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 4, x(1) = 3, x(2) = 2$, and $x(3) = 1$, evaluate its DFT $X(k)$ using the decimation-in-time FFT method, and determine the number of complex multiplications.

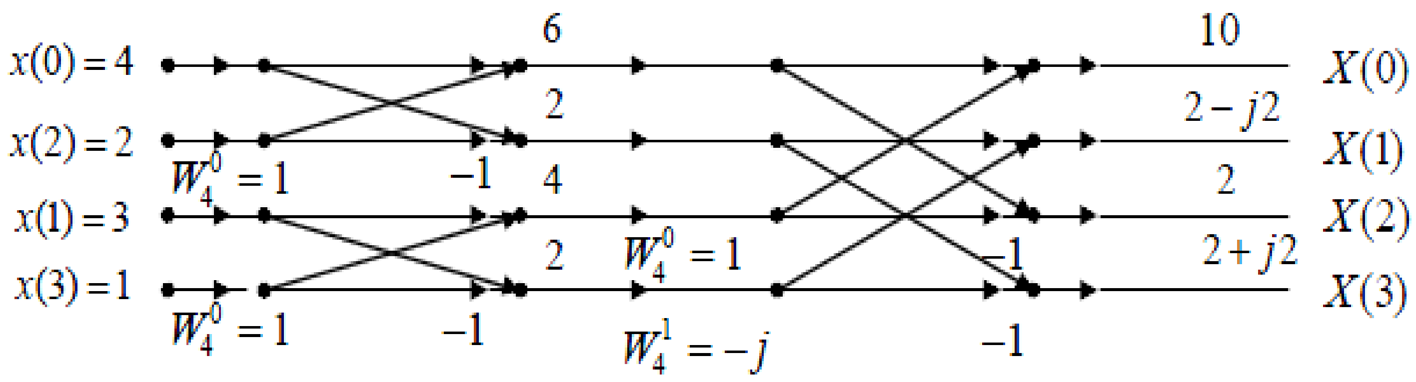
Solution 7

$$X(k) = G(k) + W_N^k H(k), \quad \text{for } k = 0, 1, \cdots, \frac{N}{2} - 1$$

$$X\left(\frac{N}{2} + k\right) = G(k) - W_N^k H(k), \quad \text{for } k = 0, 1, \cdots, \frac{N}{2} - 1$$

$$G(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} = DFT\{x(2m) \text{ with } (N/2) \text{ points}\}$$

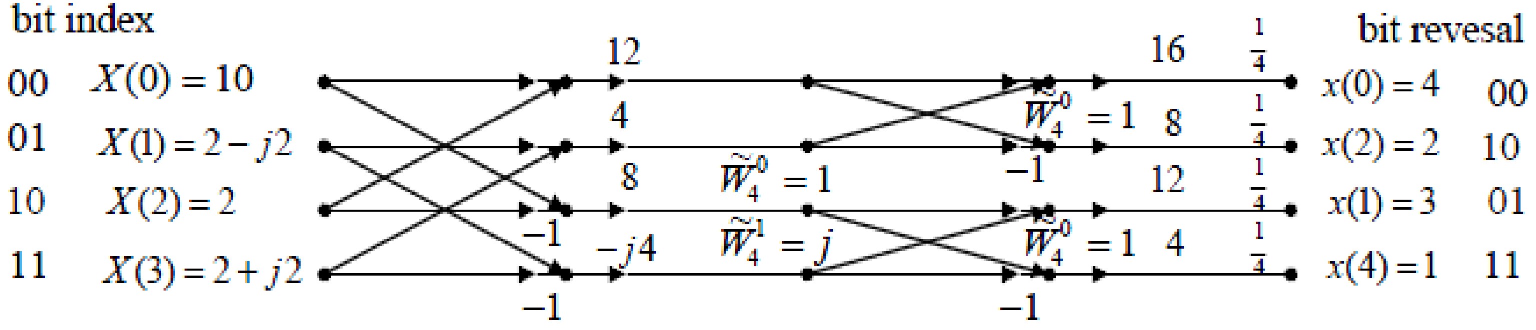
$$H(k) = \sum_{m=0}^{(N/2)-1} x(2m+1) W_{N/2}^{mk} = DFT\{x(2m+1) \text{ with } (N/2) \text{ points}\}$$



Exercise 8

Given the DFT sequence $X(k)$ for $0 \le k \le 3$ obtained in exercise 7, evaluate its inverse DFT $x(n)$ using the decimation-in-frequency FFT method.

Solution 7



$x(0)=4$, $x(1)=3$, $x(2)=2$, $x(3)=1$, 4 complex multiplications