1 Change of variables in $\mathbb{R}^n$

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Theorem 1.1
Let $(X, \mathscr{A}, \mu)$ be measure space and $(Y, \mathscr{B})$ be measurable space. Let $g\colon X \to Y$ be a measurable function. We define the measure ν on Y by:

$$\nu(B) = \mu(g^{-1}(B)) \text{ for all measurable subset } B \subseteq Y.$$

ν is called the transport measure of μ or the pullback measure of μ.
Let $f\colon Y \to \overline{\mathbb{R}}$, then

$$\int_Y f(y)d\nu(y) = \int_X (f \circ g)(x)d\mu(x).$$

Proof .
First suppose $f = \chi_B$. Let $A = g^{-1}(B) \subseteq X$. Then $f \circ g = \chi_A$, and we have

$$\int_Y f(y)d\nu(y) = \int_Y \chi_B(y)d\nu(y) = \nu(B) = \mu(g^{-1}(B)) = \mu(A) = \int_X (f \circ g)(x)d\mu(x).$$

Since both sides of this equation are linear in f, the equation holds whenever f is simple. Applying the standard procedure, the equation is then proved for all measurable function f.

Remark .
If $(X, \mathscr{A})$ be measurable space and $(Y, \mathscr{B}, \nu)$ is a measure space. Let $g\colon X \to Y$ be a bijective function and its inverse is measurable. We define the measure μ on X by: $\mu(A) = \nu(g(A))$, and it follows that

$$\int_Y f(y)d\nu(y) = \int_X (f \circ g)(x)d\mu(x).$$

2 Change of variables in $\mathbb{R}^n$

This section will be devoted to prove the Change of variables in $\mathbb{R}^n$ theorem. For this we prove two fundamental lemmas. The first called the Factorization of diffeomorphisms lemma and the second is called the volume differential lemma.

Lemma 2.1 Factorization of diffeomorphisms
Let g be a diffeomorphism of open sets in $\mathbb{R}^n$, for $n \geq 2$. For any point in the domain of g, there is a neighborhood A around that point where g can be expressed as the composition:

$$g_{\restriction A} = u \circ v$$

of a diffeomorphism u that fixes some $1 \leq m \leq n-1$ coordinates of $\mathbb{R}^n$ and another diffeomorphism v that fixes the other $n - m$ coordinates.

Proof .
We have to solve the above equation for the appropriate diffeomorphisms $v\colon A \to v(A)$ and $u\colon v(A) \to g(A)$. Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$ be the coordinate values. Labelling the first batch and second batch of coordinates with the subscripts "1" and "2" respectively, we expand:

$$\begin{aligned}
g(x,y) &= u(v(x,y)) \\
&= \Big(u_1(v_1(x,y), v_2(x,y)), u_2(v(x,y))\Big) \\
&= \Big(u_1(v_1(x,y), y), u_2(v(x,y))\Big) \text{ since } v \text{ fixes last coordinates} \\
&= \Big(v_1(x,y), u_2(v(x,y))\Big) \text{ since } u \text{ fixes first coordinates.}
\end{aligned}$$

Or, succinctly:

$$g_1 = v_1 \,, \quad g_2 = u_2 \circ v \,.$$

The first equation determines the solution function v trivially. The second equation can be inverted by the inverse function theorem:

$$u_2 = g_2 \circ v^{-1} \,,$$

for

$$\mathcal{D}v(x,y) = \begin{pmatrix} \mathcal{D}_1 g_1(x,y) & \mathcal{D}_2 g_1(x,y) \\ 0 & I \end{pmatrix}, \quad \det \mathcal{D}v(x,y) = \det \mathcal{D}_1 g_1(x,y) \neq 0 \,.$$

So given a starting point (x_0, y_0) in the domain of g, we can define v^{-1} on some open set B containing $v(x_0, y_0)$. Then take $A = v^{-1}(B)$. □

Lemma 2.2 Volume Differential
Let $g\colon U \to V$ be a diffeomorphism between open sets in $\mathbb{R}^n$. Then for all measurable subset $A \subseteq X$,

$$\lambda(g(A)) = \int_{g(A)} d\lambda(x) = \int_A |\det \mathcal{D}g(x)| d\lambda(x). \tag{2.1}$$

Proof .
First step It suffices to prove the lemma locally. That is, suppose there exists an open cover of U, $\{U_\alpha\}$, so that the equation 2.1 holds for measurable A contained inside one of the U_α. Then equation 2.1 actually holds for all measurable $A \subseteq U$.

Proof .
By passing to a countable subcover (Lindelöf's theorem), we may assume there are only countably many U_k. Define the disjoint measurable sets $E_k = U_k \setminus \bigcup_{j=1}^{k-1} U_j$, which also cover U. And define the two measures:

$$\mu(A) = \lambda(g(A)), \quad \nu(A) = \int_A |\det \mathcal{D}g(x)| d\lambda(x).$$

Now let $A \subseteq U$ be any measurable set. We have $A \cap E_j \subseteq U_j$, so $\mu(A \cap E_j) = \nu(A \cap E_j)$ by hypothesis. Therefore,

$$\mu(A) = \mu\Big(\bigcup_{j=1}^{+\infty} A \cap E_j\Big) = \sum_{j=1}^{+\infty} \mu(A \cap E_j) = \sum_{j=1}^{+\infty} \nu(A \cap E_j) = \nu(A).$$

Second step Suppose the equation 2.1 holds for two diffeomorphisms g and h, and all measurable sets. Then it holds for the composition diffeomorphism $g \circ h$, and all measurable sets.

Proof .
For any measurable A,

$$\begin{aligned}
\int_{g(h(A))} d\lambda(x) &= \int_{h(A)} |\det \mathcal{D}g(x)| d\lambda(x) \\
&= \int_A |(\det \mathcal{D}g) \circ h(x)| \cdot |\det \mathcal{D}h(x)| d\lambda(x) \\
&= \int_A |\det \mathcal{D}(g \circ h)(x)| d\lambda(x).
\end{aligned}$$

The second equality follows from 2.2 applied to the diffeomorphism h, which is valid once we know $\lambda(h(B)) = \int_B |\det \mathcal{D}h|$ for all measurable B.

Proof of the lemma .
We proceed to prove the lemma by induction, on the dimension n.

1. **Case** $n = 1$.

   Cover U by a countable set of bounded intervals I_k in $\mathbb{R}$. By the first reduction, it suffices to prove the lemma for measurable sets contained in each of the interval I_k individually. By the uniqueness of measures, it also suffices to show $\mu = \nu$ only for the intervals $[a, b]$, $[a, b[$, $]a, b]$ and $]a, b[$.

   In the case of closed interval, this is just the Fundamental Theorem of Calculus:

$$\int_{g([a,b])} d\lambda(x) = |g(b) - g(a)| = |\int_a^b g'(x)d\lambda(x)| = \int_a^b |g'(x)|d\lambda(x).$$

For the last equality, remember that g, being a diffeomorphism, must have a derivative that is positive on all of $[a, b]$ or negative on all of $[a, b]$.

If the interval is not closed, say $]a, b[$, then we may not be able to apply the Fundamental Theorem directly, but we may still obtain the desired equation by taking limits:

$$\begin{aligned}\int_{g((a,b))} d\lambda(x) &= \lim_{n\to\infty} \int_{g([a+\frac{1}{n}, b-\frac{1}{n}])} d\lambda(x) \\ &= \lim_{n\to\infty} \int_{[a+\frac{1}{n}, b-\frac{1}{n}]} |g'|(x)d\lambda(x) = \int_{(a,b)} |g'(x)|d\lambda(x).\end{aligned}$$

2. **General case.**

   According to 2.1, the diffeomorphism g can always be factored locally (i.e. on a sufficiently small open set around each point $x \in U$) as $g = h_k \circ \ldots \circ h_2 \circ h_1$, where each h_i is a diffeomorphism and fixes one coordinate of $\mathbb{R}^n$. By reduction 1, it suffices to consider this local case only. By reduction 2, it suffices to prove the lemma for each of the diffeomorphisms h_i.

   So suppose g fixes one coordinate. For convenience in notation, assume g fixes the last coordinate: $g(u, v) = (h_v(u), v)$, for $u \in \mathbb{R}^{n-1}, v \in \mathbb{R}$, and h_v are functions on open subsets of $\mathbb{R}^{n-1}$. Clearly h_v are one-to-one, and most importantly, $\det \mathcal{D}h_v(u) = \det \mathcal{D}g(u, v) \neq 0$.

   Next, let a measurable set A be given, and consider its projection and cross-section:

$$V = \{v \in \mathbb{R} : (u, v) \in A\}, \quad U_v = \{u \in \mathbb{R}^{n-1} : (u, v) \in A\}.$$

   We now apply Fubini's theorem and the induction hypothesis on the diffeomorphisms h_v:

$$\begin{aligned}\int_{g(A)} d\lambda(u, v) &= \int_{v\in V} \left(\int_{h_v(U_v)} d\lambda(u)\right) d\lambda(v) \\ &= \int_{v\in V} \left(\int_{u\in U_v} |\det \mathcal{D}h_v(u)|d\lambda(u)\right) d\lambda(v) \\ &= \int_{v\in V} \int_{u\in U_v} |\det \mathcal{D}g(u, v)|d\lambda(u, v) = \int_A |\det \mathcal{D}g(u, v)|d\lambda(u, v).\end{aligned}$$

Theorem 2.3 *Differential change of variables in* $\mathbb{R}^n$

Let $g\colon X \to Y$ be a diffeomorphism of open sets in $\mathbb{R}^n$. If $A \subseteq X$ is measurable, and $f\colon Y \to \mathbb{R}$ is measurable, then

$$\int_{g(A)} f(y)\,dy = \int_A f(g(x))\,g(dx) = \int_A f(g(x)) \cdot |\det \mathcal{D}g(x)|\,dx\,.$$

(Substitute $y = g(x)$ and $dy = g(dx) = |\det \mathcal{D}g(x)|\,dx$.)

Proof .
Take $\nu = \lambda$ and $\mu = \nu \circ g$, then

$$\int_Y f(y) d\lambda(y) = \int_X (f \circ g)(x)\, d(\lambda \circ g)(x).$$

We have $d(\lambda \circ g)(x) = |\det \mathcal{D}g(x)| d\lambda(x)$ from equation, so by 2.2

$$\int_X (f \circ g)(x)\, d(\lambda \circ g)(x) = \int_X (f \circ g)(x) \cdot |\det \mathcal{D}g(x)| d\lambda(x).$$

If we replace f by $f\,\chi_{g(A)}$, then

$$\int_{g(A)} f(x) d\lambda(x) = \int_A (f \circ g)(x) \cdot |\det \mathcal{D}g(x)| d\lambda(x).$$