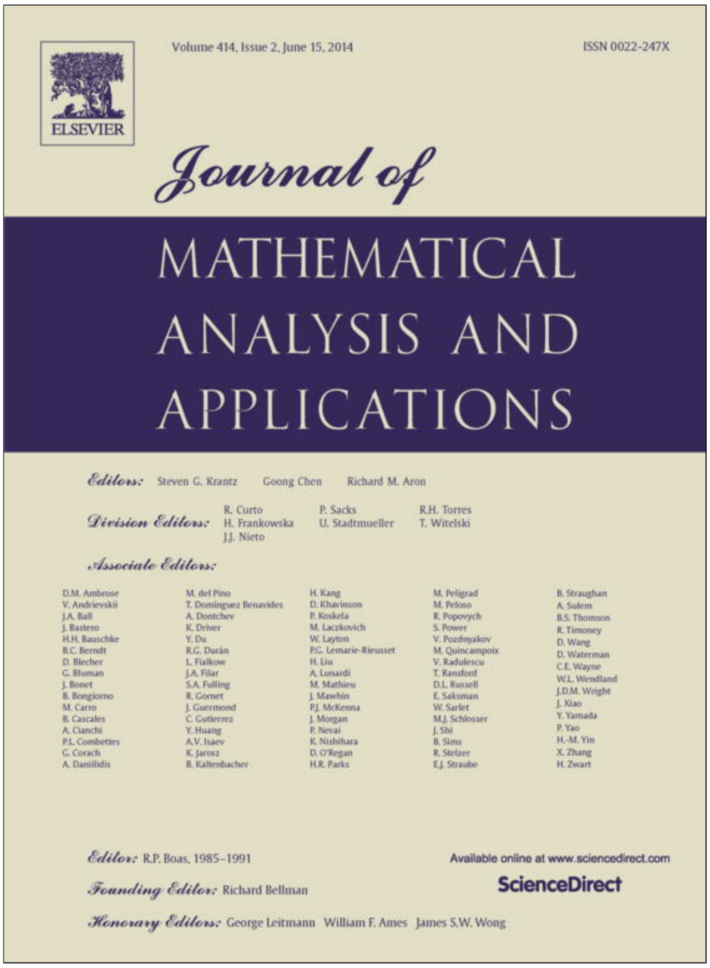
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The λ-function in JB^*-triples

Haifa M. Tahlawi, Akhlaq A. Siddiqui, Fatmah B. Jamjoom *

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455-5, Riyadh 11451, Saudi Arabia

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ABSTRACT

We discuss the λ-function in the general setting of JB^*-triples. Several results connecting the λ-function with the distance of a vector to the Brown–Pedersen's quasi-invertible elements and extreme convex decompositions have been obtained for JB^*-triples; these include JB^*-triple analogues of some related C^*-algebra results due to M. Rørdam, L. Brown and G. Pedersen.

1. Introduction

It has long been realized that the underlying structure making several interesting results on C^*-algebras hold is not the presence of an associative product xy but the presence of the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$ or the Jordan triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$ (cf. [5,16]). This provided one of the stimuli for the development of Jordan product or Jordan triple product generalizations of C^*-algebras, these including JB^*-algebras [17] and JB^*-triples [6]. Jordan analogues of various C^*-algebra results on linear isometries, extreme points and faces of the closed unit ball have been proved (cf. [5,6,9–18], and the references in [1]).

In 1987, R.M. Aron and R.H. Lohman introduced, in a motivating and celebrated paper, a geometric function, called the λ-function, determined by extreme points of the unit ball of a normed space; a normed space is said to have the λ-property if for each element x of its unit ball we have $\lambda(x) > 0$ (cf. [2]). One of the problems left open by Aron and Lohman is the following question: "What spaces of operators have the λ-property and what does the λ-function look like for these spaces?" (cf. [2]). To answer this question in the setting of C^*-algebras, Brown and Pedersen [3] introduced a notion of quasi-invertible elements in a C^*-algebra. As is well explained in [4], the Brown–Pedersen's quasi-invertible elements (in short,

* Corresponding author.
E-mail addresses: htahlawi@ksu.edu.sa (H.M. Tahlawi), asiddiqui@ksu.edu.sa (A.A. Siddiqui), fjamjoom@ksu.edu.sa (F.B. Jamjoom).

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BP-quasi-invertible elements) bear many interesting properties similar to those of invertible elements. They obtained several results verifying that the relationships between the extreme convex decomposition theory, λ-function and the distance, $\alpha_q(x)$, from a vector x to the set of BP-quasi-invertible elements are analogous with the relationships in the earlier C^*-algebra theory of unitary convex decompositions, λ_u-function and regular approximations.

Recently, we initiated a study of BP-quasi-invertible elements in the setting of JB^*-triples (cf. [13–15]). In this paper, we discuss the λ-function in the general setting of JB^*-triples. We obtain several results connecting the λ-function with $\alpha_q(x)$ and extreme convex decompositions in a JB^*-triple. We discuss closely related set-valued functions $\mathcal{V}(x)$, $\mathcal{S}(x)$ (see the next section) and obtain some estimates on $\inf \mathcal{V}(x)$ in terms of $\alpha_q(x)$. For convenience, we restrict the study of the λ-function to JB^*-triples $\mathcal{J}$ satisfying $\mathcal{E}(\mathcal{J})_1 \neq \emptyset$, where $\mathcal{E}(\mathcal{J})_1$ denotes the set of extreme points of the closed unit ball $(\mathcal{J})_1$ of $\mathcal{J}$.

Concerning the relationship between $\lambda(x)$ and $\alpha_q(x)$, we prove that

(a) $x \in \mathcal{J}_q^{-1}$ if, and only if, there exists $v \in \mathcal{E}(\mathcal{J})_1$ and a unitary u in the JB^*-algebra $\mathcal{J}_1(v)$ such that $\|x-u\| < 1$. In particular, for each $x \in \mathcal{J}_q^{-1}$ the set $\mathcal{V}(x) \cap [1,2)$ is nonempty;

(b) For each $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$, we have:
  (i) $\lambda(x) \leqslant \frac{1}{2}(1-\alpha_q(x))$;
  (ii) $\lambda(x) = 0$ when $\alpha_q(x) = 1$

(see Theorems 3.5 and 3.7 and Corollary 3.6).

Contrary to what is established by Brown and Pedersen in the setting of C^*-algebras (cf. [4]), we do not know whether the equality $\lambda(x) = \frac{1}{2}(1-\alpha_q(x))$ holds for every $x \in (\mathcal{J})_1 \backslash \mathcal{J}_q^{-1}$ or when $\lambda(x) = 0$ implies $\alpha_q(x) = 1$. In this concern, we introduce a condition in the last section, called the Λ-condition, and give some positive answers in certain cases. In the course of our analysis, we obtain JB^*-triple analogues of some other related results on C^*-algebras, due to M. Rørdam [8], L. Brown and G.K. Pedersen [3,4,7].

2. BP-quasi-invertible elements

A Jordan triple system is a vector space $\mathcal{J}$ over a field of characteristic not 2, endowed with a triple product $\{xyz\}$, which is linear and symmetric in the outer variables x, z and linear or anti-linear in the inner variable y, satisfying the Jordan identity: $\{xu\{yvz\}\} + \{\{xvy\}uz\} - \{yv\{xuz\}\} = \{x\{uyv\}z\}$ [16]. A JB^*-triple is a complex Banach space $\mathcal{J}$ together with a continuous, sesquilinear, operator-valued map $(x,y) \in \mathcal{J} \times \mathcal{J} \mapsto L(x,y)$ that defines a triple product $L(x,y)z := \{xyz\}$ in $\mathcal{J}$ making it a Jordan triple system such that each $L(x,x)$ is a positive hermitian operator on $\mathcal{J}$ and $\|\{xxx\}\| = \|x\|^3$, for all $x \in \mathcal{J}$ [16]. Thus, any JB^*-algebra is a JB^*-triple with the triple product $\{xyz\} := (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x$. A basic operator $P(x,y)$ on the JB^*-triple $\mathcal{J}$ is defined by $P(x,y)z := \{xzy\}$ for all $x, y, z \in \mathcal{J}$; we write $P(x,x)$ in short as $P(x)$. Another basic operator $B(x,y)$, called the Bergman operator, is defined on $\mathcal{J}$ by $B(x,y) := I - 2L(x,y) + P(x)P(y)$, where I is the identity operator.

As in [15], an element x in a JB^*-triple $\mathcal{J}$ is called BP-quasi-invertible with BP-quasi-inverse $y \in \mathcal{J}$ if $B(x,y) = 0$. It is known that $B(x,y) = 0 \Rightarrow B(y,x) = 0$. A BP-quasi-invertible element need not admit a unique BP-quasi-inverse; $\mathcal{J}_q^{-1}$ includes $\mathcal{E}(\mathcal{J}_1)$; and $x \in \mathcal{J}_q^{-1} \Leftrightarrow x$ is positive and invertible in the Peirce 1-space $\mathcal{J}_1(v)$ induced by some $v \in \mathcal{E}(\mathcal{J})_1$ (cf. [15]). When $\mathcal{J}$ is a JB^*-algebra, $\mathcal{J}_q^{-1}$ contains the set $\mathcal{J}^{-1}$ of all invertible elements in $\mathcal{J}$.

To each $\delta \geqslant 1$, there corresponds the following set:

$$co_\delta \mathcal{E}(\mathcal{J})_1 := \left\{ \frac{1}{\delta}\left(\sum_{i=1}^{n-1} v_i + (1+\delta-n)v_n\right):\ v_j \in \mathcal{E}(\mathcal{J})_1,\ j = 1,\ldots,n,\ n \in \mathbb{N},\ n-1 \leqslant \delta \leqslant n \right\},$$

where $\mathbb{N}$ denotes the set of positive integers.

Of course, $co_\delta \mathcal{E}(\mathcal{J})_1 \subseteq co\mathcal{E}(\mathcal{J})_1$, the convex hull of $\mathcal{E}(\mathcal{J})_1$. For each $x \in (\mathcal{J})_1$, we define $\mathcal{V}(x) := \{\beta \geqslant 1\colon x \in co_\beta \mathcal{E}(\mathcal{J})_1\}$.

Theorem 2.1. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$.*

(i) *If $\gamma \in \mathcal{V}(x)$, then $\gamma - 1 \geqslant \gamma\alpha_q(x) + 1$.*
(ii) *If $\alpha_q(x) < 1$ then $\mathcal{V}(x) \subseteq [2(1 - \alpha_q(x))^{-1}, \infty)$.*

Proof. By [13, Theorem 3.5], $x \notin \mathcal{E}(\mathcal{J})_1$ since $x \notin \mathcal{J}_q^{-1}$, and so $1 \notin \mathcal{V}(x)$.

(i) Let $\gamma \in \mathcal{V}(x)$. Then, $\gamma > 1$ and $x = \gamma^{-1} \sum_{i=1}^{n-1} v_i + \gamma^{-1}(\gamma + 1 - n)v_n$ with $v_i \in \mathcal{E}(\mathcal{J})_1$, $i = 1, \dots, n$ and $n-1 < \gamma \leqslant n$. So, $dist(\gamma x, \mathcal{E}(\mathcal{J})_1) \leqslant \|\gamma x - v_1\| = \|\sum_{i=2}^{n-1} v_i + (\gamma+1-n)v_n\| \leqslant (n-2) + (\gamma+1-n) = \gamma - 1$. Hence, by [15, Lemma 25 and Theorem 26] and the fact $\gamma > 1$, we get $\gamma - 1 \geqslant dist(\gamma x, \mathcal{E}(\mathcal{J})_1) \geqslant \max\{\alpha_q(\gamma x) + 1, \|\gamma x\| - 1\} \geqslant \alpha_q(\gamma x) + 1 = \gamma\alpha_q(x) + 1$.
(ii) If $\alpha_q(x) < 1$ and $\gamma \in \mathcal{V}(x)$, then $\gamma \geqslant 2(1 - \alpha_q(x))^{-1}$ by Part (i). Thus, $\mathcal{V}(x) \subseteq [2(1 - \alpha_q(x))^{-1}, \infty)$ by [15, Corollary 24]. □

When $x \in \mathcal{E}(\mathcal{J})_1$, we have $\mathcal{V}(x) = [1, \infty)$. So, the above result is not true in this case. Indeed, if $x \in \mathcal{J}_q^{-1} \cap (\mathcal{J})_1$ then $\alpha_q(x) = 0$ and $[2, \infty) \subseteq \mathcal{V}(x)$ by [15, Theorem 20 and Corollary 24]; however, $\mathcal{V}(x) \nsubseteq [2, \infty)$ by Corollary 3.6 below. Thus, the above result is not true when $x \in \mathcal{J}_q^{-1}$.

For each $x \in \mathcal{J}$, we define $e_m(x) := \min\{n\colon x = \frac{1}{n} \sum_{j=1}^{n} v_j, v_j \in \mathcal{E}(\mathcal{J})_1\}$; if x has no such decomposition, $e_m(x) := \infty$.

Corollary 2.2. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1$ with $\alpha_q(x) = 1$. Then, $\mathcal{V}(x) = \emptyset$ and $e_m(x) = \infty$. Moreover, $\{y \in \mathcal{J}_1(v)\colon \|y\| = \alpha_q(y) = 1\} \subseteq (\mathcal{J})_1 \backslash co\mathcal{E}(\mathcal{J})_1$.*

Proof. Since $\alpha_q(x) = 1$, $x \notin \mathcal{J}_q^{-1}$. If $\gamma \in \mathcal{V}(x)$ then $\gamma - 1 \geqslant \gamma + 1$ by Theorem 2.1; an absurdity. So, $\mathcal{V}(x) = \emptyset$. Hence, $e_m(x) > n$, for all $n \in \mathbb{N}$, by [15, Lemma 22]. Thus, $x \in (\mathcal{J}_1(v))_1 \backslash co\mathcal{E}(\mathcal{J})_1 \subseteq (\mathcal{J})_1 \backslash co\mathcal{E}(\mathcal{J})_1$. □

Corollary 2.3. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$, $x \in (\mathcal{J}_1(v))_1$ and integer $n \geqslant 2$. Then $e_m(x) \leqslant n \Rightarrow \alpha_q(x) \leqslant 1 - \frac{2}{n}$.*

Proof. If $x \in \mathcal{J}_q^{-1}$, then $\alpha_q(x) = 0 \leqslant 1 - \frac{2}{n}$. Let $x \notin \mathcal{J}_q^{-1}$ and $e_m(x) \leqslant n$, but $1 - \frac{2}{n} < \alpha_q(x)$. Then, $\alpha_q(x) < 1$: otherwise, $\alpha_q(x) = 1$ since $\alpha_q(x) \leqslant \|x\|$ by [15, Lemma 25]. So, $e_m(x) = \infty$ by Corollary 2.2; a contradiction. It follows that $n < 2(1 - \alpha_q(x))^{-1}$. Hence, $n \notin \mathcal{V}(x)$ by Theorem 2.1; this with $e_m(x) \leqslant n$ contradicts [15, Lemma 22]. □

3. The λ-function

In a JB^*-triple $\mathcal{J}$ with $\mathcal{E}(\mathcal{J})_1 \neq \emptyset$, a triple (v, y, λ) is said to be amenable to $x \in (\mathcal{J})_1$ if $v \in \mathcal{E}(\mathcal{J})_1$, $y \in (\mathcal{J})_1$ and $0 \leqslant \lambda \leqslant 1$ with $x = \lambda v + (1 - \lambda)y$. We define $\mathcal{S}(x) := \{0 \leqslant \lambda \leqslant 1\colon (v, y, \lambda)$ is amenable to $x\}$. Note that $0 \in \mathcal{S}(x)$, for every $x \in (\mathcal{J})_1$. The λ-function is defined by $x \in (\mathcal{J})_1 \mapsto \lambda(x) := \sup \mathcal{S}(x)$ (cf. [2]).

Theorem 3.1. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1$.*

(i) *If $\lambda \in \mathcal{S}(x)$ and $\lambda > 0$ then $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$.*
(ii) *If $\delta \in \mathcal{V}(x)$ then $\delta^{-1} \in \mathcal{S}(x)$.*
(iii) *$\lambda(x) = 0$ if and only if $\mathcal{V}(x) = \emptyset$.*
(iv) *If $\lambda(x) > 0$ then $\mathcal{S}(x) = [0, \lambda(x))$ or $[0, \lambda(x)]$.*

(v) *If* $\lambda(x) > 0$ *and* $0 < \gamma < \lambda(x)$ *then* $\gamma^{-1} \in \mathcal{V}(x)$.
(vi) *If* $\lambda(x) > 0$, *then* $(\inf \mathcal{V}(x))^{-1} = \lambda(x)$.
(vii) *If* $\inf \mathcal{V}(x) \in \mathcal{V}(x)$, *then* $\lambda(x) \in \mathcal{S}(x)$.

Proof. (i) $\lambda \in \mathcal{S}(x)$ means $x = \lambda w + (1-\lambda)y$ with $w \in \mathcal{E}(\mathcal{J}_1)$, $y \in (\mathcal{J})_1$. So, $\|\lambda^{-1}x - w\| \leqslant \lambda^{-1} - 1$ as $\lambda > 0$. Hence, $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$ by [15, Theorem 23].

(ii) Let $\delta \in \mathcal{V}(x)$. Then, $x = \delta^{-1}\sum_{i=1}^{n-1} v_i + \delta^{-1}(1+\delta-n)v_n$ for some $\delta \geqslant 1$ and $v_i \in \mathcal{E}(\mathcal{J})_1$, $i = 1, \dots, n$ with $n - 1 < \delta \leqslant n$. If $\delta = 1$, then $x \in \mathcal{E}(\mathcal{J})_1$, hence the result is clear as $\delta^{-1} = 1$. Next, suppose $\delta > 1$. Then $x = \delta^{-1}v_1 + \delta^{-1}(\delta-1)[\sum_{i=2}^{n-1} \frac{1}{(\delta-1)} v_i + \frac{1}{(\delta-1)}(1+\delta-n)v_n]$. Hence, $x = \delta^{-1}v_1 + (1-\delta^{-1})y$ where $y = \sum_{i=2}^{n-1} \frac{1}{(\delta-1)} v_i + \frac{1}{(\delta-1)}(1+\delta-n)v_n$ with $\|y\| \leqslant \frac{n-2}{\delta-1} + \frac{1+\delta-n}{\delta-1} = 1$, and so $\delta^{-1} \in \mathcal{S}(x)$.

(iii) This is clear from the parts (i) and (ii).

(iv) As seen above, $0 \in \mathcal{S}(x)$. Let $\lambda(x) > 0$. Then, there exists an increasing sequence (γ_n) in $\mathcal{S}(x)$ such that $\lim_{n\to\infty} \gamma_n = \lambda(x)$. Now, for any fixed $\gamma \in (0, \lambda(x))$, there exists integer N such that $\gamma_n > \gamma$ for all $n \geqslant N$. Hence, by [15, Corollary 24], $\gamma^{-1} \in (\gamma_n^{-1}, \infty) \subseteq \mathcal{V}(x)$, so that $\gamma \in \mathcal{S}(x)$ by Part (ii). However, $\lambda(x) = \sup \mathcal{S}(x)$. Thus, $\mathcal{S}(x) = [0, \lambda(x))$ or $[0, \lambda(x)]$.

(v) This is clear from the statements (ii) and (iv).

(vi) From the statement (v), we have $\lambda(x) \leqslant (\inf \mathcal{V}(x))^{-1}$. Hence, the required equality follows from the statement (ii).

(vii) Since $\inf \mathcal{V}(x) \in \mathcal{V}(x)$, $\mathcal{V}(x) \neq \emptyset$. Then, by Part (iii), $\lambda(x) > 0$. Hence, $(\inf \mathcal{V}(x))^{-1} = \lambda(x)$ by Part (vi). Thus, $\lambda(x) \in \mathcal{S}(x)$ by Part (ii). □

For any $x \in (\mathcal{J}_1(v))_1^{\circ}$ with $v \in \mathcal{E}(\mathcal{J})_1$, $\mathcal{V}(x) \neq \emptyset$ by [15, Theorem 16 and Lemma 22], and so $\lambda(x) \neq 0$ by Theorem 3.1.

Corollary 3.2. *Let* $\mathcal{J}$ *be a* JB^**-triple. Then, for any fixed* $v \in \mathcal{E}(\mathcal{J})_1$ *and* $x \in (\mathcal{J}_1(v))_1$, *the following assertions are equivalent*:

(i) $\alpha_q(x) < 1$ *implies* $\mathcal{V}(x) \neq \emptyset$.
(ii) $\lambda(x) = 0$ *implies* $\alpha_q(x) = 1$.
(iii) $\alpha_q(x) < 1$ *implies* $\lambda(x) > 0$.

Proof. If $x \in \mathcal{J}_q^{-1}$, then $\alpha_q(x) = 0$, $\mathcal{V}(x) \neq \emptyset$ and $\lambda(x) \neq 0$. Thus the equivalences hold true by [15, Theorem 20 and Corollary 24] (see comments before Corollary 2.2). Next, suppose $x \notin \mathcal{J}_q^{-1}$. Then:

(i) ⇒ (ii): If $\lambda(x) = 0$, $\mathcal{V}(x) = \emptyset$ by Theorem 3.1, and hence $\alpha_q(x) \geqslant 1$ by (i). But $\alpha_q(x) \leqslant \|x\| = 1$. Therefore, $\alpha_q(x) = 1$.

(ii) ⇒ (iii): If $\alpha_q(x) < 1$, $\lambda(x) \neq 0$ by (ii), and hence $\lambda(x) > 0$.

(iii) ⇒ (i): $\alpha_q(x) < 1$ with $\mathcal{V}(x) = \emptyset$ gives $\lambda(x) = 0$ by Theorem 3.1. □

Corollary 3.3. *For a* JB^**-triple* $\mathcal{J}$ *with* $v \in \mathcal{E}(\mathcal{J})_1$ *and* $x \in (\mathcal{J}_1(v))_1 \cap \mathcal{J}_q^{-1}$:

(i) $dist(x, \mathcal{E}(\mathcal{J})_1) \leqslant 1$.
(ii) $\lambda(x) \geqslant \frac{1}{2}$.
(iii) $\mathcal{S}(x) \neq \emptyset$ *and* $\mathcal{V}(x) \neq \emptyset$.

Proof. (i) By [15, Theorem 20], $v_1, v_2 \in \mathcal{E}(\mathcal{J})_1$ with $x = \frac{1}{2}(v_1 + v_2)$. Hence, $dist(x, \mathcal{E}(\mathcal{J})_1) = \inf_{v\in\mathcal{E}(\mathcal{J})_1} \|x - v\| \leqslant \|x - v_1\| = \|\frac{1}{2}(v_2 - v_1)\| \leqslant 1$.

(ii) Clear from the definition of λ-function and Part (i).

(iii) As seen above, $\frac{1}{2} \in \mathcal{S}(x)$, and so $(2, \infty) \subseteq \mathcal{V}(x)$ by Theorem 3.1. □

The following result is essentially the same as [15, Theorem 9]:

Theorem 3.4. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))^{-1}$. Then there exists $u \in \mathcal{U}(\mathcal{J}_1(v))$ such that x is positive and invertible in the unitary isotope $\mathcal{J}_1(v)^{[u]}$, which coincides with the Peirce 1-space $\mathcal{J}_1(u)$. Moreover, $x \in \mathcal{J}_q^{-1}$. Here, $(\mathcal{J}_1(v))^{-1}$ denotes the set of invertible elements in $\mathcal{J}_1(v)$.*

If $v \in \mathcal{E}(\mathcal{J})_1$, then for every element $x \in (\mathcal{J}_1(v))_1$, we define the set $T_v(x) := \{0 \leqslant \lambda \leqslant 1\colon x = \lambda u + (1-\lambda)y$ with $u \in \mathcal{U}(\mathcal{J}_1(v)), y \in (\mathcal{J}_1(v))_1\}$, where $\mathcal{U}(\mathcal{J}_1(v))$ denotes the set of unitary elements in the Peirce 1-space $\mathcal{J}_1(v)$ of $\mathcal{J}$ induced by v. Thus, $T_v(x) \subseteq \mathcal{S}(x)$ by [10, Lemma 4].

Theorem 3.5. *Let $\mathcal{J}$ be a JB^*-triple and $x \in (\mathcal{J})_1$. Then, the following assertions are equivalent:*

(i) $x \in \mathcal{J}_q^{-1}$.
(ii) $x \in \delta\mathcal{U}(\mathcal{J}_1(v)) + (1-\delta)\mathcal{U}(\mathcal{J}_1(v))$ *for some* $v \in \mathcal{E}(\mathcal{J})_1$ *and* $0 \leqslant \delta < \frac{1}{2}$.
(iii) $1-\delta \in T_v(x)$ *for some* $v \in \mathcal{E}(\mathcal{J})_1$ *and* $0 \leqslant \delta < \frac{1}{2}$.
(iv) $dist(x, \mathcal{U}(\mathcal{J}_1(v))) \leqslant 2\delta$ *for some* $v \in \mathcal{E}(\mathcal{J})_1$ *and* $0 \leqslant \delta < \frac{1}{2}$.

Proof. (i) $\Rightarrow$ (ii): Let $x \in \mathcal{J}_q^{-1}$. Then, by [15, Theorem 11], there is some $v \in \mathcal{E}(\mathcal{J})_1$ such that x is positive and invertible in the Peirce 1-space $\mathcal{J}_1(v)$, which is a JB^*-algebra with unit v. Hence, by the continuous functional calculus of JB^*-algebras, $\sigma_{\mathcal{J}_1(v)}(x) \subseteq [-1,1] \backslash (2\delta-1, 1-2\delta)$ for some $0 \leqslant \delta < \frac{1}{2}$ since $0 \notin \sigma_{\mathcal{J}_1(v)}(x)$. Moreover, by [11, Lemma 2.4], we get $x \in \delta\mathcal{U}(\mathcal{J}_1(v)) + (1-\delta)\mathcal{U}(\mathcal{J}_1(v))$.

(ii) $\Rightarrow$ (iii): $x = \delta v_1 + (1-\delta)v_2$ $(= (1-\delta)v_2 + \delta v_1)$ with $v_1, v_2 \in \mathcal{U}(\mathcal{J}_1(v))$ gives $(1-\delta) \in T_v(x)$ since v_1 being a unitary in $\mathcal{J}_1(v)$ has norm 1.

(iii) $\Rightarrow$ (iv): If $x = (1-\delta)w + \delta y$ with $w \in \mathcal{U}(\mathcal{J}_1(v))$ and $y \in (\mathcal{J})_1$ and for some $0 \leqslant \delta < \frac{1}{2}$, then $\|x - w\| \leqslant 2\delta < 1$, and hence $dist(x, \mathcal{U}(\mathcal{J}_1(v))) < 1$.

(iv) $\Rightarrow$ (ii): Clear from [15, Theorem 21].

(ii) $\Rightarrow$ (i): $x \in (\mathcal{J}_1(v))^{-1}$ by (ii). Hence, $x \in \mathcal{J}_q^{-1}$ by Theorem 3.4. □

Corollary 3.6. *If $x \in \mathcal{J}_q^{-1}$ then $\lambda \in \mathcal{V}(x)$ for some $1 \leqslant \lambda < 2$.*

Proof. Follows straightforwardly from Theorem 3.5. □

Next result gives an upper bound of $\lambda(x)$ for $x \notin \mathcal{J}_q^{-1}$ in terms of $\alpha_q(x)$.

Theorem 3.7. *Let $\mathcal{J}$ be a JB^*-triple with $v \in \mathcal{E}(\mathcal{J})_1$. Then, for any $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$, we have:*

(a) $\lambda(x) \leqslant \frac{1}{2}(1 - \alpha_q(x))$.
(b) *If* $\alpha_q(x) = 1$ *then* $\lambda(x) = 0$.

Proof. (a) Under the hypothesis, we have $dist(x, \mathcal{E}(\mathcal{J})_1) \geqslant \max\{\alpha_q(x) + 1, \|x\| - 1\}$, by [15, Theorem 26]. Now, if $x = \lambda w + (1-\lambda)y$, where $w \in \mathcal{E}(\mathcal{J})_1$, $y \in (\mathcal{J})_1$ and $0 \leqslant \lambda \leqslant 1$, we have $x - w = (\lambda - 1)w + (1-\lambda)y$ and so $\alpha_q(x) + 1 = dist(x, \mathcal{E}(\mathcal{J})_1) \leqslant \|x - w\| = |1-\lambda|\|y - w\| \leqslant 2(1-\lambda)$; which gives $\lambda \leqslant \frac{1}{2}(1 - \alpha_q(x))$. This proves the Part (a).

(b) Immediate from the Part (a) since $\lambda(x) \geqslant 0$. □

Corollary 3.8. *For a JB^*-triple $\mathcal{J}$ with $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1^{\circ} \backslash \mathcal{J}_q^{-1}$:*

(i) $\mathcal{V}(x) \neq \emptyset$.
(ii) $\mathcal{V}(x) = [(\lambda(x))^{-1}, \infty)$ *or* $\mathcal{V}(x) = ((\lambda(x))^{-1}, \infty)$.

(iii) $e_m(x) = n$ *if* $n \neq (\lambda(x))^{-1}$ *given by* $n - 1 < (\lambda(x))^{-1} \leqslant n$.
(iv) $e_m(x) = n$ *or* $e_m(x) = n + 1$ *if* $n = (\lambda(x))^{-1}$.

Proof. (i) By [15, Theorem 16], for any $x \in (\mathcal{J}_1(v))_1^\circ$, we have $e_m(x) < \infty$, and hence $\mathcal{V}(x) \neq \emptyset$ by [15, Lemma 22].

(ii) Since $\mathcal{V}(x) \neq \emptyset$, $e_m(x) = (\inf \mathcal{V}(x))^{-1}$ by Theorem 3.1(vi). This together with [15, Corollary 24] proves the Part (ii).

The other parts follow easily from the Part (ii) and Theorem 3.7. □

4. The Λ-condition

Let $\mathcal{J}$ be a JB^*-triple. For any $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$ with $v \in \mathcal{E}(\mathcal{J})_1$ and $\alpha_q(x) = 1$, we have $\lambda(x) = 0$ by Theorem 3.7. To get more progress on the λ-function, we introduce the Λ-condition on $\mathcal{J}$, as follows:

$$x \in \big(\mathcal{J}_1(v)\big)_1 \backslash \mathcal{J}_q^{-1} \quad \text{with } v \in \mathcal{E}(\mathcal{J})_1 \quad \text{and} \quad \lambda(x) = 0 \quad \Rightarrow \quad \alpha_q(x) = 1.$$

With the Λ-condition, $(\mathcal{J}_1(v))_1 \backslash co\mathcal{E}(\mathcal{J})_1 \subseteq \{y \in \mathcal{J}_1(v)\colon \|y\| = \alpha_q(y) = 1\}$ by [15, Theorem 16] and Theorem 3.1; hence, $(\mathcal{J}_1(v))_1 = co\mathcal{E}(\mathcal{J})_1$ if $\alpha_q(x) < 1$. Let $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$ and $\alpha_q(x) < 1$. Of course, $\lambda(x) > 0$. If $\|x\| = 1$ then $\mathcal{V}(x) \neq \emptyset$ by Corollary 3.2, and so $\mathcal{V}(x) = [(\lambda(x))^{-1}, \infty)$ or $\mathcal{V}(x) = ((\lambda(x))^{-1}, \infty)$ by [15, Corollary 24] and Theorem 3.1 (compare Corollary 3.8). Hence, for $n - 1 < (\lambda(x))^{-1} \leqslant n$, $\lambda(x) = n$ if $n \neq (\lambda(x))^{-1}$; and given by $e_m(x) = n$ or $e_m(x) = n + 1$ if $n = (\lambda(x))^{-1}$. In either case, for every $0 < \epsilon \leqslant 1$, we have $x = (\epsilon + n)^{-1}(v_1 + \cdots + v_n + \epsilon v_{n+1})$ with $v_1, \ldots, v_{n+1} \in \mathcal{E}(\mathcal{J})_1$.

Now, we see if one can identify $\inf \mathcal{V}(x)$ in terms of $\alpha_q(x)$. To each $x \in \mathcal{J} \backslash \mathcal{J}_q^{-1}$ with $\alpha_q(x) < 1$, we associate the number $\beta_{q_x} := 2(1 - \alpha_q(x))^{-1}$.

Theorem 4.1. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$ with $\alpha_q(x) < 1$. Then, the following conditions are equivalent*:

(Λ_1) $(\beta_{q_x}, \infty) \subseteq \mathcal{V}(x)$.
(Λ_2) $(\lambda(x))^{-1} = \inf \mathcal{V}(x) = \beta_{q_x}$.
(Λ_3) *For all $\gamma > \beta_{q_x}$, there is $w \in \mathcal{E}(\mathcal{J})_1$ with $\|\gamma x - w\| \leqslant \gamma - 1$.*
(Λ_4) $\lambda(x) \geqslant \beta_{q_x}^{-1}$.

Proof. (Λ_1) ⇒ (Λ_2): By Theorem 2.1, $\mathcal{V}(x) \subseteq [\beta_{q_x}, \infty)$. So, $\inf \mathcal{V}(x) = \beta_{q_x}$ by the condition (Λ_1); the required equality follows from Theorem 3.1.

(Λ_2) ⇒ (Λ_3): See [15, Theorem 23].

(Λ_3) ⇒ (Λ_4): Let $\gamma > \beta_{q_x}$. Then, by (Λ_3), there is $w \in \mathcal{E}(\mathcal{J})_1$ with $\|\lambda x - w\| \leqslant \lambda - 1$. By Theorem 3.1, $(\gamma, \infty) \subseteq \mathcal{V}(x)$, so that $\inf \mathcal{V}(x) \leqslant \gamma$. Hence, $\lambda(x) \geqslant \gamma^{-1}$ by Theorem 3.1. Thus, $\lambda(x) \geqslant \beta_{q_x}^{-1}$.

(Λ_4) ⇒ (Λ_1): Let $\gamma > \beta_{q_x}$. Then, $0 < \gamma^{-1} < \beta_{q_x}^{-1} \leqslant \lambda(x)$ by (Λ_4). Thus, $\gamma^{-1} \in \mathcal{S}(x)$, so $(\gamma, \infty) \subseteq \mathcal{V}(x)$ by Theorem 3.1. □

If $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$ with $\|x\| = 1$ and $\alpha_q(x) < 1$, then $rx \in (\mathcal{J})_1$ and $\alpha_q(rx) = r\alpha_q(x) < 1$, for each $0 < r \leqslant 1$, by [15, Lemma 25]. We conclude this article with the following result on vectors of norm 1:

Theorem 4.2. *Let $\mathcal{J}$ be a JB^*-triple with an extreme point v of $(\mathcal{J})_1$ and let $x \in (\mathcal{J}_1(v))_1 \backslash \mathcal{J}_q^{-1}$ with $\|x\| = 1$ and $\alpha_q(x) < 1$.*

(a) *The following assertions are equivalent*:
   (i) (Λ_2) *holds for* x.
   (ii) (Λ_2) *holds for all* rx *with* $0 < r \leqslant 1$.
   (iii) *If* $y \in Sp(x)$ *(the linear span of* x*) and* $\|y\| > \alpha_q(y)+2$*, then* $\|y-v\| \leqslant \|y\|-1$ *for some* $v \in \mathcal{E}(\mathcal{J})_1$.
(b) *If any one of the assertions* (i) *to* (iii) *holds for all unit vectors* $y \in \mathcal{J}_1(v)\backslash\mathcal{J}_q^{-1}$ *with* $\alpha_q(y) < 1$*, then* $\mathcal{J}$ *satisfies the* Λ*-condition.*

Proof. (a) (i) $\Leftrightarrow$ (ii): The implication (ii) $\Rightarrow$ (i) is clear. Conversely, suppose $(\lambda(x))^{-1} = \inf \mathcal{V}(x) = \beta_{q_x}$ and r is any fixed number such that $0 < r < 1$. Then $rx \in \mathcal{J}_1(v)_1^{\circ}\backslash\mathcal{J}_q^{-1}$, and hence $\lambda(rx) \leqslant \beta_{q_{rx}}^{-1}$ by Theorem 3.7. Let $\lambda > \beta_{q_x}$. By the assertion (i) and Theorem 2.1, $\lambda \in \mathcal{V}(x)$, so that $x \in co_\lambda\mathcal{E}(\mathcal{J})_1$. Then, $x = \lambda^{-1}(v_1 + \cdots + v_{n-1} + (1+\lambda-n)v_n)$ for some $v_1, v_2, \ldots, v_n \in \mathcal{E}(\mathcal{J})_1$ with the positive integer satisfying $n-1 < \lambda \leqslant n$. So, $rx = r\lambda^{-1}(v_1 + \cdots + v_{n-1} + (1+\lambda-n)v_n) + \frac{1-r}{2}(-v_1) + \frac{1-r}{2}v_1$. Hence, $\lambda(rx) \geqslant r\lambda^{-1} + \frac{1-r}{2} = r\beta_{q_x}^{-1} + \frac{1-r}{2} + r\lambda^{-1} - r\beta_{q_x}^{-1} = \frac{1}{2}(1-r\alpha_q(x)) + r(\lambda^{-1} - \beta_{q_x}^{-1}) = \beta_{q_{rx}}^{-1} + r(\lambda^{-1} - \beta_{q_x}^{-1})$. Therefore, $\lambda(rx) \geqslant \beta_{q_{rx}}^{-1} + r(\lambda^{-1} - \beta_{q_x}^{-1})$ for all $\lambda > \beta_{q_x}$. Thus, $\lambda(rx) = \beta_{q_{rx}}^{-1}$.

(ii) $\Rightarrow$ (iii): Under the hypothesis of (iii), $\|y\|^{-1} < \frac{1}{2}(1 - \alpha_q(\|y\|^{-1}y))$. Then, by the assertion (ii), $\|y\|^{-1} < \frac{1}{2}(1-\alpha_q(x)) \leqslant \lambda(x)$ since $x = \|y\|^{-1}y$. Now, with $\lambda = \|y\|^{-1}$, [15, Theorem 5.3] provides the existence of elements $v \in \mathcal{E}(\mathcal{J})_1$ and $b \in (\mathcal{J})_1$ such that $x = \lambda v + (1-\lambda)b$. Hence, $\|x - \lambda v\| \leqslant 1-\lambda$ as $\lambda \leqslant 1$ (in fact, $\lambda \leqslant \frac{1}{2}$ as $\lambda = \|y\|^{-1} < \frac{1}{\alpha_q(x)+2} \leqslant \frac{1}{2}$). Thus, $\|y - v\| \leqslant \|y\| - 1$.

(iii) $\Rightarrow$ (i): For any $x \in \mathcal{J}_1(v)$ with $\|x\| = 1$ and $\gamma > 2(1-\alpha_q(x))^{-1}$, we have $\|\gamma x\| - \alpha_q(\gamma x) = \gamma(1-\alpha_q(x)) > 2$, so that $\|\gamma x\| > \alpha_q(\gamma x) + 2$. Then, $\|\gamma x - v\| \leqslant \|\gamma x\| - 1$ for some $v \in \mathcal{E}(\mathcal{J})_1$ by the assertion (iii). Hence, $(\gamma, \infty) \subseteq \mathcal{V}(x)$ by [15, Theorem 23]. Thus, $(\beta_x, \infty) \subseteq \mathcal{V}(x)$.

(b) Finally, suppose $x \in \mathcal{J}_1(v)\backslash\mathcal{J}_q^{-1}$ with $\|x\| = 1$ and $\lambda(x) = 0$. Then, $\alpha_q(x) = 1$: for otherwise, $\alpha_q(x) < 1$ would give $\lambda(x) \neq 0$ by the assertion (i) and Theorem 4.1, a contradiction. $\square$

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References

[1] M. Apazoglou, A.M. Peralta, Linear isometries between real JB^*-triples and C^*-algebras, Quart. J. Math. (2013), arXiv:1309.3838, September 2013.
[2] R.M. Aron, R.H. Lohman, A geometric function determined by extreme points of the unit ball of a normed space, Pacific J. Math. 127 (1987) 209–231.
[3] L.G. Brown, G.K. Pedersen, On the geometry of the unit ball of a C*-algebra, J. Reine Angew. Math. 469 (1995) 113–147.
[4] L.G. Brown, G.K. Pedersen, Approximation and convex decomposition by extremals in C^*-algebras, Math. Scand. 81 (1997) 69–85.
[5] L.A. Harris, A generalization of C^*-algebras, Proc. Lond. Math. Soc. (3) 42 (1981) 331–361.
[6] W. Kaup, A Riemann Mapping Theorem for bounded symmetric domains in complex Banach spaces, Math. Z. 183 (1983) 503–529.
[7] G.K. Pedersen, λ-function in operator algebras, J. Operator Theory 26 (1991) 345–381.
[8] M. Rørdam, Advances in the theory of unitary rank and regular approximations, Ann. of Math. 128 (1988) 153–172.
[9] A.A. Siddiqui, JB^*-algebras of topological stable rank 1, Int. J. Math. Math. Sci. 2007 (2007), http://dx.doi.org/10.1155/2007/37186, Article ID 37186, 24 pp.
[10] A.A. Siddiqui, Average of two extreme points in JBW^*-triples, Proc. Japan Acad. Ser. A Math. Sci. 83 (2007) 176–178.
[11] A.A. Siddiqui, Convex combinations of unitaries in JB^*-algebras, New York J. Math. 17 (2011) 127–137.
[12] A.A. Siddiqui, Computation of the λ_u-function in JB^*-algebras, New York J. Math. 19 (2013) 145–158.
[13] H.M. Tahlawi, A.A. Siddiqui, Analogue of Brown–Pedersen quasi invertibility for JB*-triples, Int. J. Math. Anal. 30 (5) (2011) 1469–1476.
[14] H.M. Tahlawi, A.A. Siddiqui, On non-degenerate Jordan triple systems, Int. J. Algebra 22 (5) (2011) 1099–1105.

[15] H.M. Tahlawi, A.A. Siddiqui, F.B. Jamjoom, On the geometry of the unit ball of a JB^*-triple, Abstr. Appl. Anal. 2013 (2013), http://dx.doi.org/10.1155/2013/891249, Article ID 891249, 8 pp.
[16] H. Upmeier, Symmetric Banach Manifolds and Jordan C*-algebras, Elsevier Science Publishers B.V., 1985.
[17] J.D.M. Wright, Jordan C*-algebras, Michigan Math. J. 24 (1977) 291–302.
[18] J.D.M. Wright, M.A. Youngson, On isometries of Jordan algebras, J. Lond. Math. Soc. (2) 17 (1978) 339–344.