Some Continuous Probability Distributions: Part I

Continuous Uniform distribution

Normal Distribution

Exponential Distribution

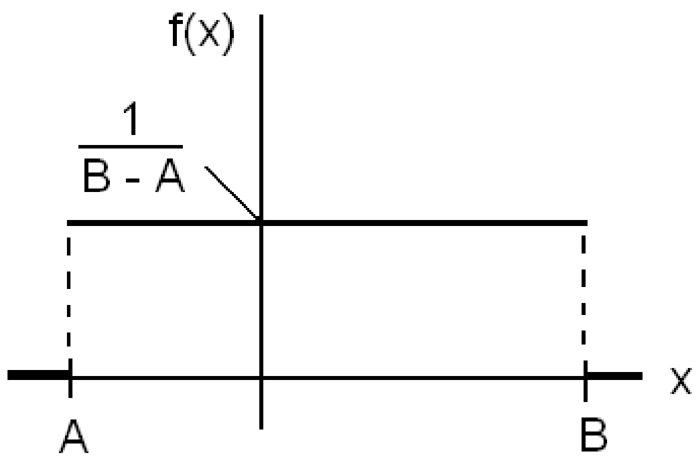
Chapter 6: Some Continuous Probability Distributions:

6.1 Continuous Uniform distribution:

(Rectangular Distribution)

The probability density function of the continuous uniform random variable X on the interval [A, B] is given by:

$$f(x) = f(x; A, B) = \begin{cases} \dfrac{1}{B-A} & ; \; A \le x \le B \\ 0 & ; \; elsewhere \end{cases}$$



We write X~Uniform(A,B).

Theorem 6.1:

The mean and the variance of the continuous uniform distribution on the interval $[A, B]$ are:

$$\mu = \frac{A+B}{2} \qquad \sigma^2 = \frac{(B-A)^2}{12}$$

Example 6.1:

Suppose that, for a certain company, the conference time, X, has a uniform distribution on the interval [0,4] (hours).

(a) What is the probability density function of X?

(b) What is the probability that any conference lasts at least 3 hours?

Solution:

$$(a) f(x) = f(x;0,4) = \begin{cases} \frac{1}{4} & ; \ 0 \le x \le 4 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

$$(b) p(X \ge 3) = \int_3^4 f(x)\,dx = \int_3^4 \frac{1}{4}\,dx = \frac{1}{4}$$

Moment generating function

The moment generating function of a uniform random variable X is defined for any $t \in \mathbb{R}$:

$$M_X(t) = \begin{cases} \frac{1}{(u-l)t}[\exp(tu) - \exp(tl)] & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Using the definition of moment generating function:

$$\begin{aligned} M_X(t) &= \mathrm{E}[\exp(tX)] \\ &= \int_{-\infty}^{\infty} \exp(tx) f_X(x) dx \\ &= \int_l^u \exp(tx) \frac{1}{u-l} dx \\ &= \frac{1}{u-l} \left[\frac{1}{t} \exp(tx) \right]_l^u \\ &= \frac{\exp(tu) - \exp(tl)}{(u-l)t} \end{aligned}$$

Note that the above derivation is valid only when $t \neq 0$. However, when $t = 0$:

$$M_X(0) = \mathrm{E}[\exp(0 \cdot X)] = \mathrm{E}[1] = 1$$

Furthermore, it is easy to verify that:

$$\lim_{t \to 0} M_X(t) = M_X(0)$$

When $t \neq 0$, the integral above is well-defined and finite for any $t \in \mathbb{R}$. Thus, the moment generating function of a uniform random variable exists for any $t \in \mathbb{R}$.

Exercise : Uniform Distribution

6.20 Given a continuous uniform distribution, show that

(a) $\mu = \frac{A+B}{2}$, and

(b) $\sigma^2 = \frac{(B-A)^2}{12}$

Solution

6.20 $f(x) = \frac{1}{B-A}$ for $A \le x \le B$.

(a) $\mu = \int_A^B \frac{x}{B-A}\,dx = \frac{B^2-A^2}{2(B-A)} = \frac{A+B}{2}$.

(b) $E(X^2) = \int_A^B \frac{x^2}{B-A}\,dx = \frac{B^3-A^3}{3(B-A)}$.

So, $\sigma^2 = \frac{B^3-A^3}{3(B-A)} - \left(\frac{A+B}{2}\right)^2 = \frac{4(B^2+AB+A^2)-3(B^2+2AB+A^2)}{12} = \frac{B^2-2AB+A^2}{12} = \frac{(B-A)^2}{12}$.

6.21 The daily amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random variable X having a continuous uniform distribution with $A = 7$ and $B = 10$. Find the probability that on a given day the amount of coffee dispensed by this machine will be

(a) at most 8.8 liters;

(b) more than 7.4 liters but less than 9.5 liters;

(c) at least 8.5 liters.

Solution

6.21 $A = 7$ and $B = 10$.

(a) $P(X \le 8.8) = \frac{8.8-7}{3} = 0.60$.

(b) $P(7.4 < X < 9.5) = \frac{9.5-7.4}{3} = 0.70$.

(c) $P(X \ge 8.5) = \frac{10-8.5}{3} = 0.50$.

6.22 A bus arrives every 10 minutes at a bus stop. It is assumed that the waiting time for a particular individual is a random variable with a continuous uniform distribution.

(a) What is the probability that the individual waits more than 7 minutes?

(b) What is the probability that the individual waits between 2 and 7 minutes?

Solution

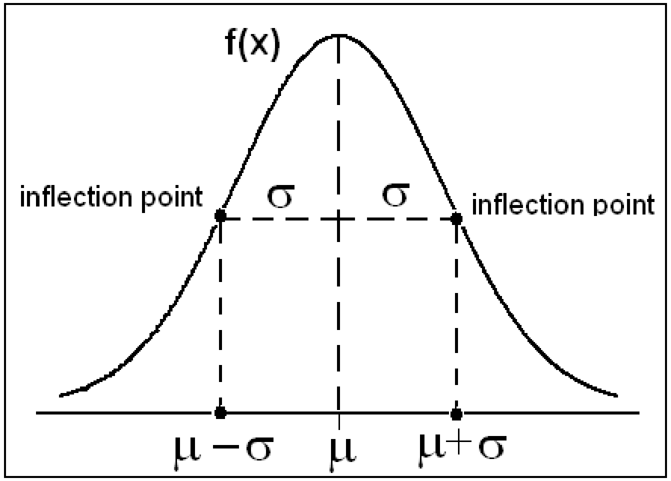
6.22 (a) $P(X > 7) = \frac{10-7}{10} = 0.3.$

(b) $P(2 < X < 7) = \frac{7-2}{10} = 0.5.$

6.2 Normal Distribution:

- The normal distribution is one of the most important continuous distributions.

- Many measurable characteristics are normally or approximately normally distributed, such as, height and weight.



- The graph of the probability density function (pdf) of a normal distribution, called the normal curve, is a bell-shaped curve.

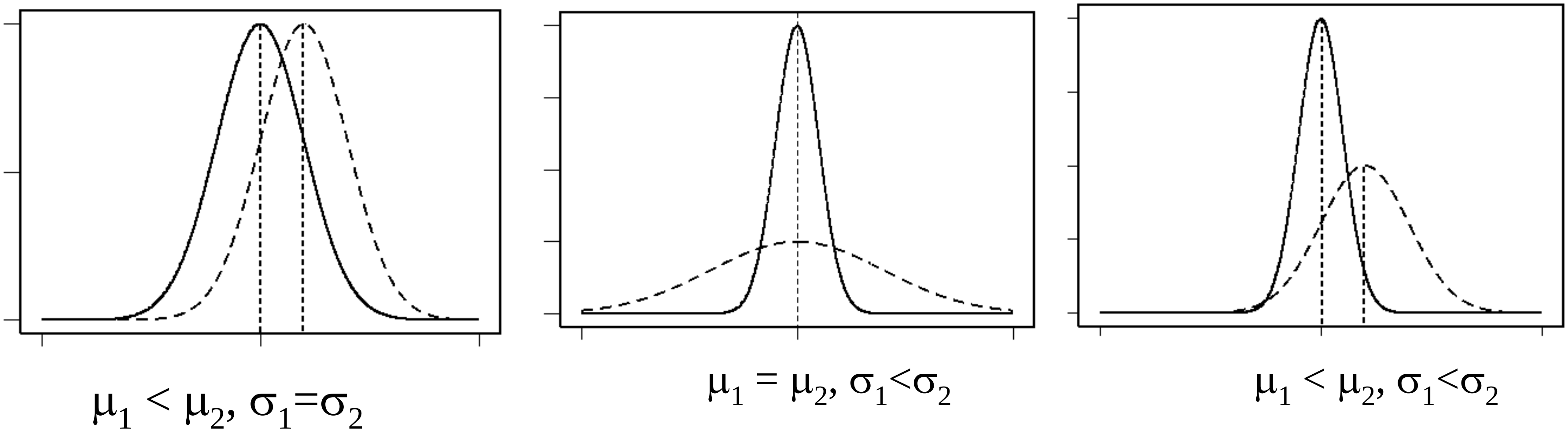
- The pdf of the normal distribution depends on two parameters: mean = E(X)= μ and variance =Var(X) = σ^2.
- If the random variable X has a normal distribution with mean μ and variance σ^2, we write:

X ~ Normal(μ,σ) or X ~ N(μ,σ)

- The pdf of X ~ Normal(μ,σ) is given by:

$$f(x) = n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \begin{cases} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{cases}$$

■ The location of the normal distribution depends on μ and its shape depends on σ.



- **Some properties of the normal curve *f*(*x*) of N(μ,σ):**

1. f(x) is symmetric about the mean μ.
2. The total area under the curve of f(x) =1.
3. The highest point of the curve of f(x) at the mean μ.

4. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $X=\mu$, (Mode = Median = Mean).

5. The curve has its points of inflection at $X=\mu \pm\sigma$ is concave downward if $\mu -\sigma < X < \mu +\sigma$ and is concave upward otherwise.

6. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.

6.3 Areas Under the Normal Curve:

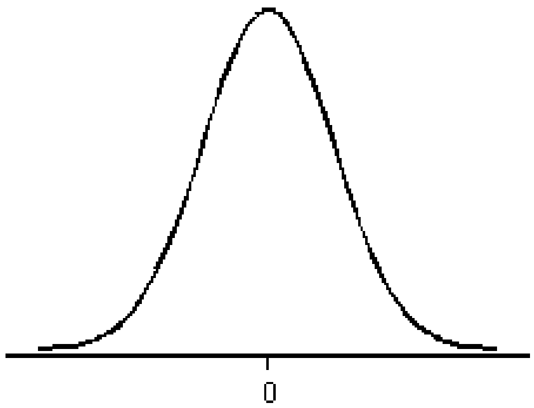
Definition 6.1:

The Standard Normal Distribution:

•The normal distribution with mean $\mu=0$ and variance $\sigma^2=1$ is called the standard normal distribution and is denoted by Normal(0,1) or N(0,1). If the random variable Z has the standard normal distribution, we write Z~Normal(0,1) or Z~N(0,1).

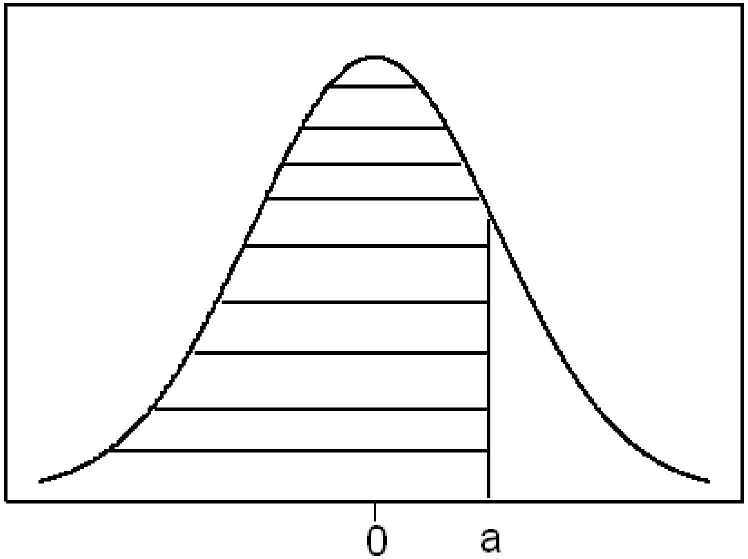
•The pdf of Z~N(0,1) is given by:

$$f(z) = n(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



•The standard normal distribution, Z~N(0,1), is very important because probabilities of any normal distribution can be calculated from the probabilities of the standard normal distribution.

•Probabilities of the standard normal distribution Z~N(0,1) of the form P(Z≤a) are tabulated (Table A.3, p681).



$$P(Z \le a) = \int_{-\infty}^{a} f(z)dz$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

= from the table

How to transform normal distribution (X) to standard normal distribution (Z)?

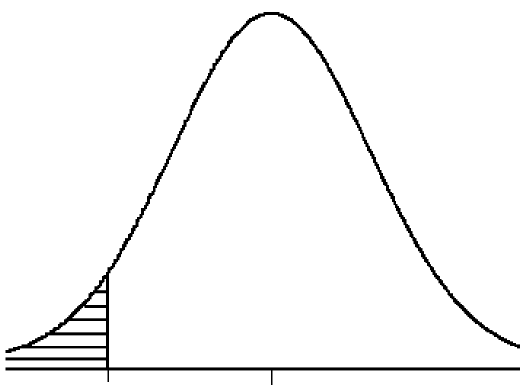
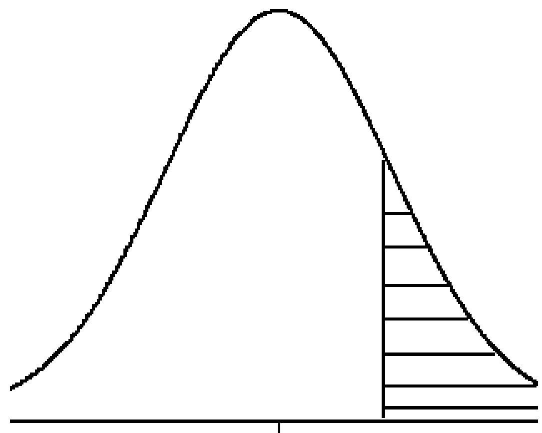
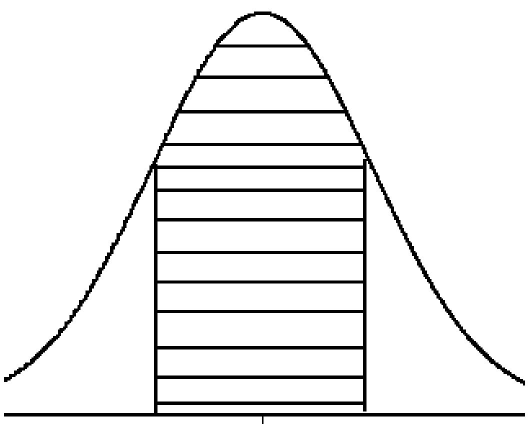
- We can transfer any normal distribution $X \sim N(\mu,\sigma)$ to the standard normal distribution, $Z \sim N(0,1)$ by using the following result.

Result: If $X \sim N(\mu,\sigma)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

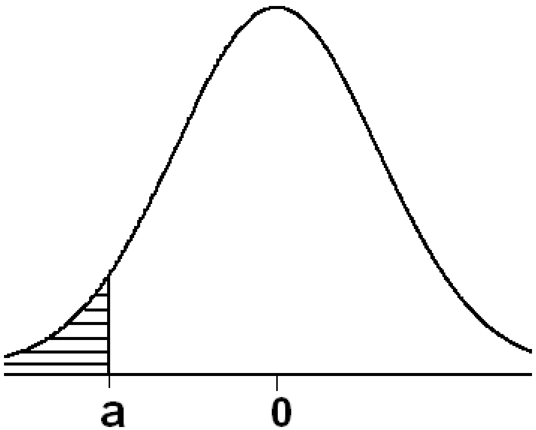
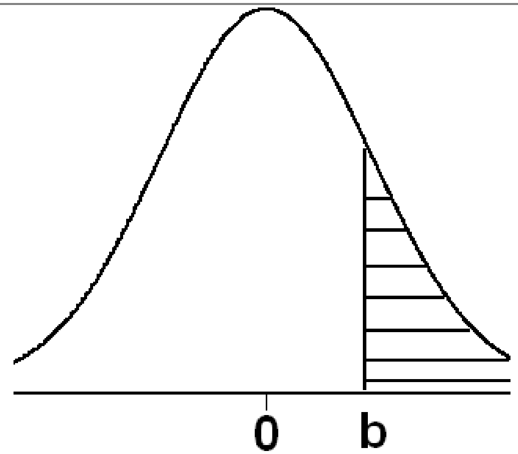
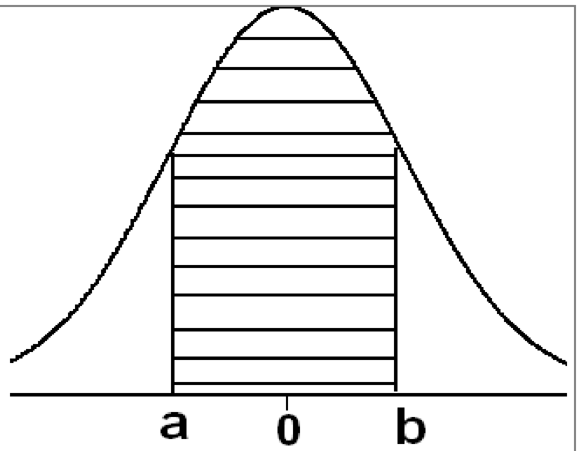
Areas Under the Normal Curve of $X \sim N(\mu,\sigma)$

The probabilities of the normal distribution $N(\mu,\sigma)$ depends on μ and σ.

		
$P(X < a) = \int_{-\infty}^{a} f(x)\,dx$	$P(X > b) = \int_{b}^{\infty} f(x)\,dx$	$P(a < X < b) = \int_{a}^{b} f(x)\,dx$

Probabilities of Z~N(0,1):

Suppose Z ~ N(0,1).

		
$P(Z \leq a)$ **The area to the left of Z**	**$P(Z \geq b) = 1 - P(Z \leq b)$** **The area to the right of Z**	**$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a)$**

Note: $P(Z=a)=0$ for every a .

“How to use tables of Z”

Example:

Suppose Z~N(0,1).

(1) P(Z≤1.50)=0.9332

Z	0.00	0.01	…
:	⇓		
1.5 ⇒	0.9332		
:			

(2) P(Z≥0.98)
=1–P(Z≤0.98)
=1 – 0.8365
= 0.1635

Z	0.00	…	0.08
:	:	:	⇓
:	…	…	⇓
0.9⇒	⇒	⇒	0.8365

(3) P(–1.33≤ Z≤2.42)
= P(Z≤2.42)– P(Z≤–1.33)
= 0.9922 – 0.0918
= 0.9004

Z	…	0.02	0.03
:	:	⇓	⇓
–1.3	⇒		0.0918
:		⇓	
2.4	⇒	0.9922	

(4) P(Z≤0)=P(Z ≥0)=0.5

Notations about Z table

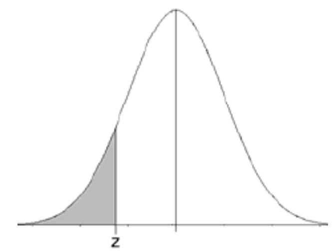
P(Z<-3.49) ≈ 0

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009							
-3.0	0.0013	0.0013	0.0013							
-2.9	0.0019	0.0018	0.0018							
-2.8	0.0026	0.0025	0.0024							
-2.7	0.0035	0.0034	0.0033							
-2.6	0.0047	0.0045	0.0044							
-2.5	0.0062	0.0060	0.0059							
...										
...										
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255							
0.4	0.6554	0.6591	0.6628							
...										
...										
...										
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Direction of increasing Of Z

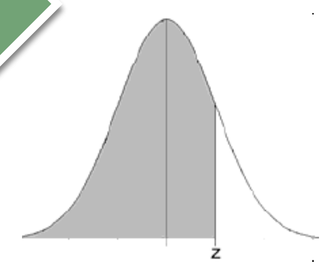
Direction of increasing probability

All probabilities for Z < 0 (negative) are less than 0.5



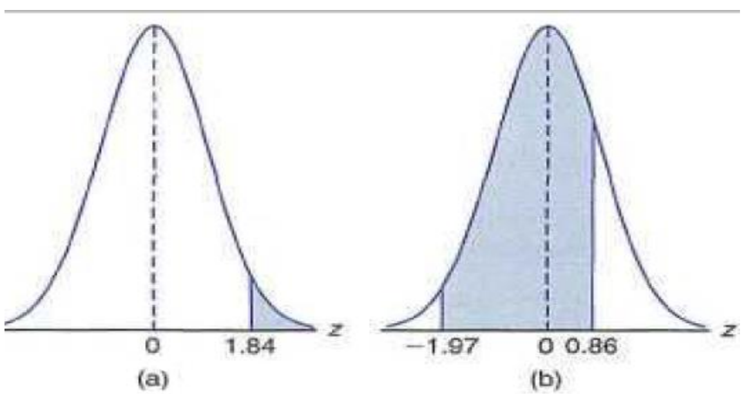
Direction of increasing probability

All probabilities for Z > 0 (positive) are more than 0.5



P(Z> 3.49) ≈ 1

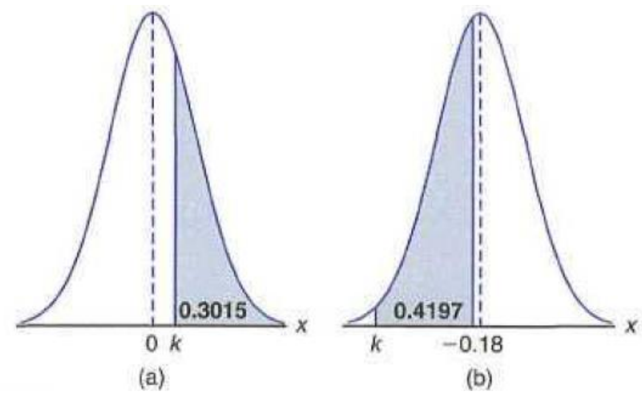
Example 6.2: Given a standard normal distribution, find the area under the curve that lies (a) to the right of z = 1.84, and (b) between z = -1.97 and z = 0.86.



Solution:

- The area in Figure (a) to the right of $z = 1.84$ is equal to 1 minus the area in Table A.3 to the left of $Z = 1.84$, namely, $1 - 0.9671 = 0.0329$.
- The area in Figure (b) between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$. From Table A.3 we find the desired area to be $0.8051 - 0.0244 = 0.7807$

Example 6.3: Given a standard normal distribution, find the value of A such that (a) *P(Z > k)* = 0.3015, and (b) *P(k < Z < -0.18)* = 0.4197.



Solution:

(a) In Figure 6.10(a) we see that the A: value leaving an area of 0.3015 to the right must then leave an area of 0.6985 to the left. From Table A.3 it follows that $k = 0.52$.

(b) From Table A.3 we note that the total area to the left of —0.18 is equal to 0.4286. In Figure 6.10(b) we see that the area between k and -0.18 is 0.4197 so that the area to the left of k must be $0.4286 - 0.4197 = 0.0089$. Hence, from Table A.3, we have $k = -2.37$. J

Example:

Suppose Z~N(0,1). Find the value of k such that $P(Z\leq k) = 0.0207$.

Solution:

$.k = -2.04$

Z	…	0.04	
:	:	⇑ ⇑	
−2.0	⇐⇐	0.0207	
:			

Example 6.2: Reading assignment

Example 6.3: Reading assignment

Probabilities of X~N(μ,σ):

- *Result:* $X \sim N(\mu,\sigma) \Leftrightarrow Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$
- $X \leq a \Leftrightarrow \dfrac{X-\mu}{\sigma} \leq \dfrac{a-\mu}{\sigma} \Leftrightarrow Z \leq \dfrac{a-\mu}{\sigma}$

1) $P(X \le a) = P\left(Z \le \frac{a-\mu}{\sigma}\right)$

2) $P(X \ge a) = 1 - P(X \le a) = 1 - P\left(Z \le \frac{a-\mu}{\sigma}\right)$

3) $P(a \le X \le b) = P(X \le b) - P(X \le a) = P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$

4) $P(X=a)=0$ for every a.

5) $P(X \le \mu) = P(X \ge \mu) = 0.5$

Example 6.4: Reading assignment
Example 6.5: Reading assignment
Example 6.6: Reading assignment

Example:

Suppose that the hemoglobin level for healthy adults males has a normal distribution with mean $\mu=16$ and variance $\sigma^2=0.81$ (standard deviation $\sigma=0.9$).

(a) Find the probability that a randomly chosen healthy adult male has hemoglobin level less than 14.

(b) What is the percentage of healthy adult males who have hemoglobin level less than 14?

Solution:

Let X = the hemoglobin level for a healthy adult male

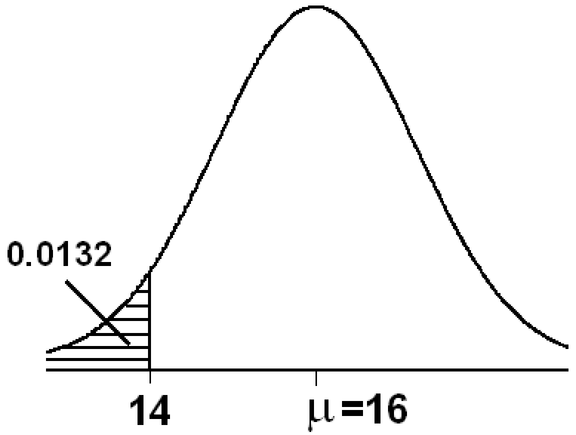
$X \sim N(\mu,\sigma) = N(16, 0.9)$.

(a) $P(X \leq 14) = P\left(Z \leq \frac{14-\mu}{\sigma}\right) = P\left(Z \leq \frac{14-16}{0.9}\right) = P(Z \leq -2.22) = 0.0132$

(b) The percentage of healthy adult males who have hemoglobin level less than 14 is

$P(X \leq 14) \times 100\% = 0.01320 \times 100\% = 1.32\%$

Therefore, 1.32% of healthy adult males have hemoglobin level less than 14.



Example:

Suppose that the birth weight of Saudi babies has a normal distribution with mean μ=3.4 and standard deviation σ=0.35.

(a) Find the probability that a randomly chosen Saudi baby has a birth weight between 3.0 and 4.0 kg.

(b) What is the percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg?

Solution:

X = birth weight of a Saudi baby

$\mu = 3.4 \quad \sigma = 0.35 \quad (\sigma^2 = 0.35^2 = 0.1225)$

$X \sim N(3.4, 0.35)$

(a) $P(3.0<X<4.0)=P(X<4.0)-P(X<3.0)$

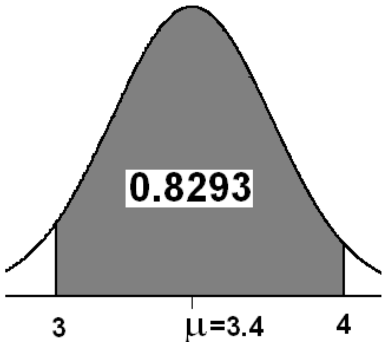
$$= P\left(Z \le \frac{4.0-\mu}{\sigma}\right) - P\left(Z \le \frac{3.0-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{4.0-3.4}{0.35}\right) - P\left(Z \le \frac{3.0-3.4}{0.35}\right)$$

$= P(Z\le1.71) - P(Z \le -1.14)$

$= 0.9564 - 0.1271$

$= 0.8293$



(b) The percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg is

$P(3.0<X<4.0) \times 100\% = 0.8293 \times 100\% = 82.93\%$

Notation:

$P(Z \leq Z_A) = A$

Result:

$Z_A = -Z_{1-A} \iff Z_{1-A} = -Z_A$

Example:

$Z \sim N(0,1)$

$P(Z \leq Z_{0.025}) = 0.025$

$P(Z \geq Z_{0.95}) = 0.25$

$P(Z \geq Z_{0.90}) = 0.10$

Example:

$Z \sim N(0,1)$

$Z_{0.025} = -Z_{0.975} = -1.96$

$Z_{0.95} = 1.645$

$Z_{0.90} = 1.285$

Z	…	0.06	
:	:	⇑ ⇑	
1.9	⇐⇐	0.975	

Example 6.10: Gauges are use to reject all components where a certain dimension is not within the specifications 1.50±d. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.20. Determine the value d such that the specifications cover 95% of the measurements.

Solution: **μ=1.5, σ=0.20, X= measurement, X~N(1.5, 0.20)**

The specification limits are: $1.5 \pm d$

x_1=Lower limit=1.5–d, x_2=Upper limit=1.5+d

$P(X > 1.5+d) = 0.025 \Leftrightarrow P(X < 1.5+d) = 0.975$

$P(X < 1.5-d) = 0.025$

$$\Leftrightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{(1.5-d)-1.5}{0.20}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

$$\Leftrightarrow \frac{-d}{0.20} = -1.96$$

$$\Leftrightarrow -d = (0.20)(-1.96)$$

$$\Leftrightarrow d = 0.392$$

Z	…	0.06	
:	:	⇑ ⇑	
-1.9	⇐⇐	0.025	

$$P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

$$\frac{-d}{0.20} = -1.96$$

$$\text{Note: } \frac{-d}{0.20} = Z_{0.025}$$

The specification limits are:
x_1=Lower limit=1.5–d = 1.5 – 0.392 = 1.108
x_2=Upper limit=1.5+d=1.5+0.392= 1.892
Therefore, 95% of the measurements fall within the specifications (1.108, 1.892).

Moment Generating Function, Mean and Variance of Normal Distribution

$$M_Z(t) = E(e^{tZ}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} \times e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+\frac{1}{2}t^2} \times e^{-\frac{1}{2}(z^2-2tz+t^2)} dz = \frac{e^{+\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} dz = e^{\frac{1}{2}t^2}$$

Because the integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} dz = 1$ as it is that of normal with mean t and variance 1.

$M_Z(t) = e^{\frac{1}{2}t^2}$. Using theorem if $Y = a + bX, \Rightarrow M_Y(t) = e^{bt} M_X(t)$. We get

we have since $Z = \frac{X - \mu}{\sigma}, \Rightarrow X = \mu + \sigma Z \Rightarrow M_X(t) = e^{\mu t} M_Z(\sigma t) = e^{\mu t} \times e^{\frac{1}{2}(\sigma t)^2}$

OR $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

THE MEAN*is* : $\mu_1' = \frac{d}{dt} M_X(t) = \frac{d}{dt} e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0} = (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0} = \mu$

AND SECOND MOMENT IS:

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) = \frac{d}{dt}\{(\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0}$$

$$= (\mu + \sigma^2 t)^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0} + (\sigma^2)e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0} = (\mu + 0)^2 \times 1 + \sigma^2 \times 1 = \mu^2 + \sigma^2$$

Variance is : $V(X) = \mu_2' - (\mu_1')^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Normal Approximation to Binomial

Theorem 6.2: .

If X is a binomial random variable with mean $p = np$ and variance $a^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to$ oo, is the standard normal distribution $n(z; 0, 1)$.

1- Determine the lower or upper limit.
2- Using + ½ with upper limit or –½ with lower.

$P(Z \leq a) = \varphi\ (a)$

- $P(X \leq b) = p(Z < \frac{b - np + \frac{1}{2}}{\sqrt{npq}}) = \varphi\left(\frac{b - np + \frac{1}{2}}{\sqrt{npq}}\right)$
- $P(X < b) = p(X \leq b-1) = p(Z < \frac{(b-1) - np + \frac{1}{2}}{\sqrt{npq}}) = \varphi\left(\frac{b - np - \frac{1}{2}}{\sqrt{npq}}\right)$
- $P(a \leq X) = p(Z > \frac{a - np - \frac{1}{2}}{\sqrt{npq}}) = 1 - \varphi\left(\frac{a - np - \frac{1}{2}}{\sqrt{npq}}\right)$
- $P(X > a) = p(X \geq a+1) = p(Z > \frac{(a+1) - np - \frac{1}{2}}{\sqrt{npq}}) = 1 - \varphi\left(\frac{a - np + \frac{1}{2}}{\sqrt{npq}}\right)$
- $P(X = c) = P(c - \frac{1}{2} \leq X \leq c + \frac{1}{2})$ ⇒ lower limit= upper limit

Example of Approximating binomial

•Let X be the number of times that a fair coin when flipped 40 times lands on head. Find the prob. that it will be equal to 20.

Solution

$$p = P(H) = \frac{1}{2}, q = \frac{1}{2}, n = 40, x = 20, np = 20, npq = 10$$

$$p(x = 20) = P(19.5 \le X \le 20.5) = P\left\{\frac{19.5 - 20}{\sqrt{10}} \le \frac{X - 20}{\sqrt{10}} \le \frac{20.5 - 20}{\sqrt{10}}\right\}$$

$$P\left\{-.16 \le \frac{X - 20}{\sqrt{10}} \le +0.16\right\} = P\{-.16 \le Z \le +0.16\} = \varphi(0.16) - \varphi(0.16) \cong 0.1272$$

$$\text{Using Binomial } P(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \approx 0.1254$$

Exercise

6.1

Given a standard normal distribution, find the normal curve area under the curve which lies

(a) to the left of a = 1.43;

(b) to the right of z = -0.89:

(c) between z = —2.16 and z = — 0.65;

(d) to the left of z = -1.39;

(e) to the: right of z = 1.90:

(f) between z = -0.48 and z = 1.74.

Solution

6.1 (a) Area=0.9236.

(b) Area=$1 - 0.1867 = 0.8133$.

(c) Area=$0.2578 - 0.0154 = 0.2424$.

(d) Area=0.0823.

(e) Area=$1 - 0.9750 = 0.0250$.

(f) Area=$0.9591 - 0.3156 = 0.6435$.

6.2 Find the value of z if the area under a standard normal curve

(a) to the right, of z is 0.:3022;
(b) to the left of z is 0.1131
(c) between 0 and z, with $z > 0$, is 0.4838;
(d) between $-z$ and z, with $z > 0$, is 0.9500.

Solution

6.2 (a) The area to the left of z is $1 - 0.3622 = 0.6378$ which is closer to the tabled value 0.6368 than to 0.6406. Therefore, we choose $z = 0.35$.

(b) From Table A.3, $z = -1.21$.

(c) The total area to the left of z is 0.5000+0.4838=0.9838. Therefore, from Table A.3, $z = 2.14$.

6.5 Given the normally distributed variable X with mean 18 and standard deviation 2.5, find

(a) $P(X < 15)$;

(b) the value of k such that $P(X < k) = 0.2236$;

(c) the value of k such that $P(X > k) = 0.1814$;

(d) $P(17 < X < 21)$.

Solution

6.5 (a) $z = (15 - 18)/2.5 = -1.2$; $P(X < 15) = P(Z < -1.2) = 0.1151$.

(b) $z = -0.76$, $k = (2.5)(-0.76) + 18 = 16.1$.

(c) $z = 0.91$, $k = (2.5)(0.91) + 18 = 20.275$.

(d) $z_1 = (17 - 18)/2.5 = -0.4$, $z_2 = (21 - 18)/2.5 = 1.2$;
$P(17 < X < 21) = P(-0.4 < Z < 1.2) = 0.8849 - 0.3446 = 0.5403$.

6.9 A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) what fraction of the cups will contain more than 224 milliliters?

(b) what is the probability that a cup contains between 191 and 209 milliliters?

(c) how many cups will probably overflow if 230-milliliter cups are used for the next 1000 drinks?

(d) below what value do we get the smallest 25% of the drinks?

Solution

6.9 (a) $z = (224 - 200)/15 = 1.6$. Fraction of the cups containing more than 224 millimeters is $P(Z > 1.6) = 0.0548$.

(b) $z_1 = (191 - 200)/15 = -0.6$, $Z_2 = (209 - 200)/15 = 0.6$; $P(191 < X < 209) = P(-0.6 < Z < 0.6) = 0.7257 - 0.2743 = 0.4514$.

(c) $z = (230 - 200)/15 = 2.0$; $P(X > 230) = P(Z > 2.0) = 0.0228$. Therefore, $(1000)(0.0228) = 22.8$ or approximately 23 cups will overflow.

(d) $z = -0.67$, $x = (15)(-0.67) + 200 = 189.95$ millimeters.

Approximation to the binomial

6.24 A coin is tossed 400 times. Use the normal-curve approximation to find the probability of obtaining

(a) between 185 and 210 heads inclusive;

(b) exactly 205 heads;

(c) less than 176 or more than 227 heads.

Solution

6.24 $\mu = np = (400)(1/2) = 200$, $\sigma = \sqrt{npq} = \sqrt{(400)(1/2)(1/2)} = 10$.

(a) $z_1 = (184.5 - 200)/10 = -1.55$ and $z_2 = (210.5 - 200)/10 = 1.05$.
$P(184.5 < X < 210.5) = P(-1.55 < Z < 1.05) = 0.8531 - 0.0606 = 0.7925$.

(b) $z_1 = (204.5 - 200)/10 = 0.45$ and $z_2 = (205.5 - 200)/10 = 0.55$.
$P(204.5 < X < 205.5) = P(0.45 < Z < 0.55) = 0.7088 - 0.6736 = 0.0352$.

(c) $z_1 = (175.5 - 200)/10 = -2.45$ and $z_2 = (227.5 - 200)/10 = 2.75$.
$P(X < 175.5) + P(X > 227.5) = P(Z < -2.45) + P(Z > 2.75)$
$= P(Z < -2.45) + 1 - P(Z < 2.75) = 0.0071 + 1 - 0.9970 = 0.0101$.

6.26 A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives

(a) exceeds 13?

(b) is less than 8?

Solution

6.26 $\mu = np = (100)(0.1) = 10$ and $\sigma = \sqrt{(100)(0.1)(0.9)} = 3$.

(a) $z = (13.5 - 10)/3 = 1.17$; $P(X > 13.5) = P(Z > 1.17) = 0.1210$.

(b) $z = (7.5 - 10)/3 = -0.83$; $P(X < 7.5) = P(Z < -0.83) = 0.2033$.

6.34 A pair of dice is rolled 180 times. What is the probability that a total of 7 occurs

(a) at least 25 times?

(b) between 33 and 41 times inclusive?

(c) exactly 30 times?

Solution

6.34 $\mu = (180)(1/6) = 30$ and $\sigma = \sqrt{(180)(1/6)(5/6)} = 5$.

(a) $z = (24.5 - 30)/5 = -1.1$; $P(X > 24.5) = P(Z > -1.1) = 1 - 0.1357 = 0.8643$.

(b) $z_1 = (32.5 - 30)/5 = 0.5$ and $z_2 = (41.5 - 30)/5 = 2.3$.
$P(32.5 < X < 41.5) = P(0.5 < Z < 2.3) = 0.9893 - 0.6915 = 0.2978$.

(c) $z_1 = (29.5 - 30)/5 = -0.1$ and $z_2 = (30.5 - 30)/5 = 0.1$.
$P(29.5 < X < 30.5) = P(-0.1 < Z < 0.1) = 0.5398 - 0.4602 = 0.0796$.

6.35 A company produces component parts for an engine. Parts specifications suggest that 95% of items meet specifications. The parts are shipped to customers in lots of 100.

(a) What is the probability that more than 2 items will be defective in a given lot?

(b) What is the probability that more than 10 items will be defective in a lot?

Solution

6.35 (a) $p = 0.05$, $n = 100$ with $\mu = 5$ and $\sigma = \sqrt{(100)(0.05)(0.95)} = 2.1794$.
So, $z = (2.5 - 5)/2.1794 = -1.147$; $P(X \geq 2) \approx P(Z \geq -1.147) = 0.8749$.

(b) $z = (10.5 - 5)/2.1794 = 2.524$; $P(X \geq 10) \approx P(Z > 2.52) = 0.0059$.

Some Continuous Probability Distributions:

Exponential Distribution

Exponential Distribution

- Usually, exponential distribution is used to describe the time or distance until some event happens.
- It is in the form of:

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

  - where $x \geq 0$ and $\mu > 0$. μ is the mean or expected value.

$$e = 2.71828$$

Another form of exponential distribution

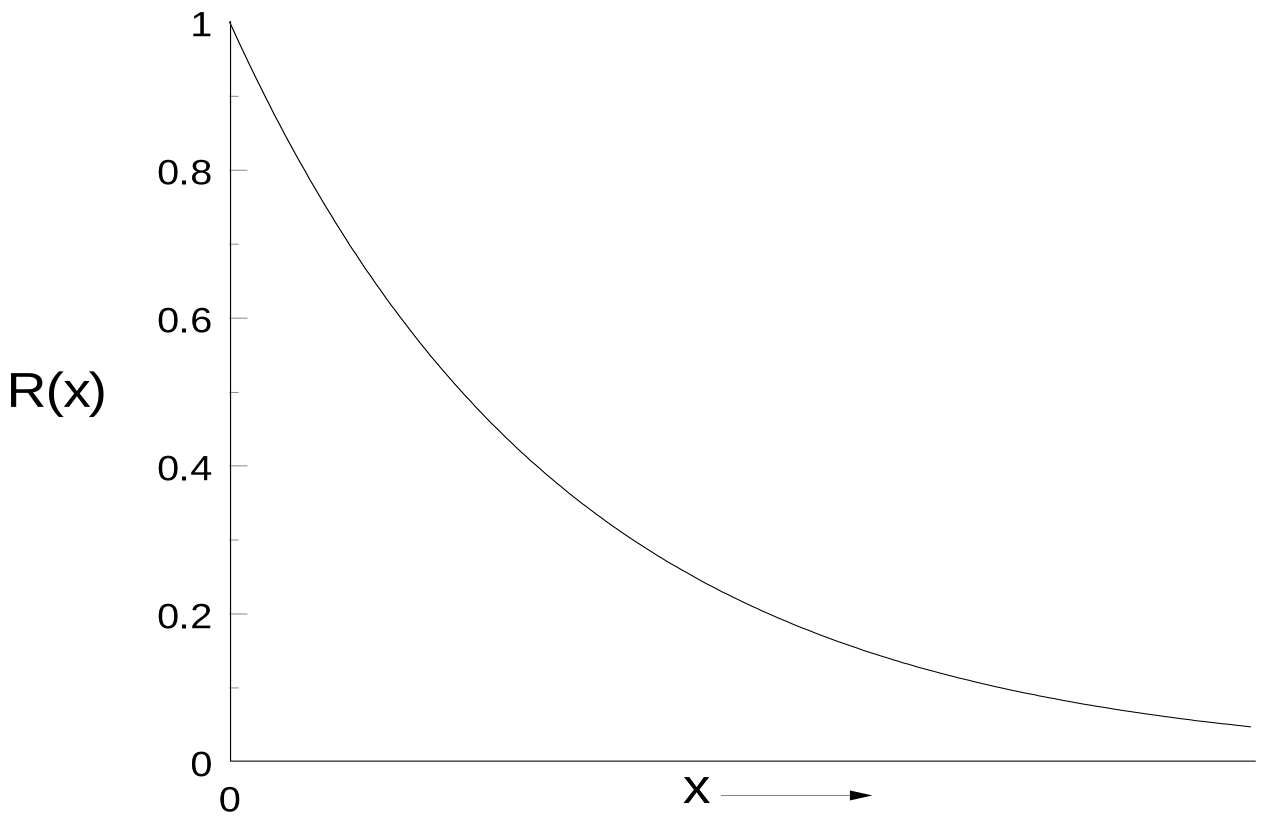
$$f(x) = \lambda e^{-\lambda x}$$

In this case, $\lambda = \frac{1}{\mu}$

Then the mean or expected value is $\frac{1}{\lambda}$

- For example, the exponential random variable is used to measure the waiting time for elevator to come.
- The exponential distribution has a number of useful applications. For example, we can use it to describe arrivals at a car wash or the time it takes to load a truck.

What should exponential distribution look like



Mean and variance of exponential distribution

- E(X)= μ or $\frac{1}{\lambda}$
- Var(X)= μ^2 or $\frac{1}{\lambda^2}$

Cumulative probabilities

$$P(x \le x_0) = 1 - e^{-x_0/\mu}$$

Where x_0 is some specific value of x

How to find probability?

- We use CDF to find probabilities under exponential distribution.

$$P(x \le x_0) = \int_0^{x_0} f(x)dx = 1 - e^{-\frac{x_0}{\mu}}$$

- Or

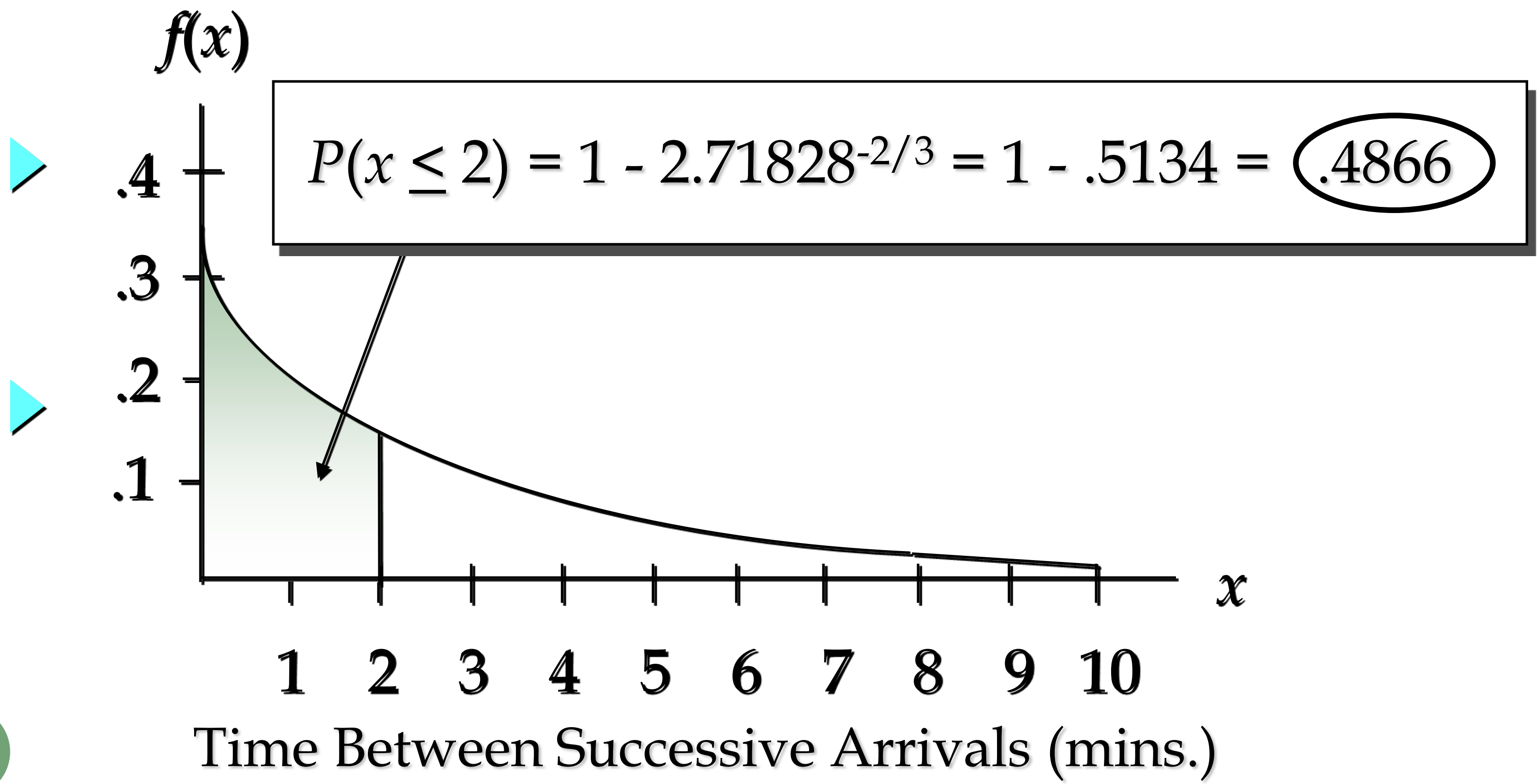
$$P(x_a \le x \le x_b) = \int_{x_a}^{x_b} f(x)dx = P(x \le x_b) - P(x \le x_a)$$

$$= \int_{x_a}^{x_b} f(x)dx = \int_0^{x_b} f(x)dx - \int_0^{x_a} f(x)dx = 1 - e^{-\frac{x_b}{\mu}} - (1 - e^{-\frac{x_a}{\mu}}) = e^{-\frac{x_a}{\mu}} - e^{-\frac{x_b}{\mu}}$$

Example: Al's Full-Service Pump

The time between arrivals of cars at Al's full-service gas pump follows an exponential probability distribution with a mean time between arrivals of 3 minutes. Al would like to know the probability that the time between two successive arrivals will be 2 minutes or less.

$$P(x \leq 2) = 1 - 2.71828^{-2/3} = 1 - .5134 = .4866$$



Lack of memory

- That is a very interesting and useful property for exponential distribution.
- It is called "**Memorylessness**" or simply "**Lack of memory**".
- In mathematical form: $P(X > s+t \mid X > s) = P(X > t)$
- Therefore, P(wait more than 10 minutes| wait more than 3 minutes)=P(wait more than 7+3 minutes| wait more than 3 minutes)=P(wait more than 7 minutes)

Example

- On average, it takes about 5 minutes to get an elevator at stat building. Let X be the waiting time until the elevator arrives. (Let's use the form with μ here)
  - Find the pdf of X.
  - What is the probability that you will wait less than 3 minutes?
  - What is the probability that you will wait for more than 10 minutes?
  - What is the probability that you will wait for more than 7 minutes?
  - Given that you already wait for more than 3 minutes, what is the probability that you will wait for more than 10 minutes?

Relationship between the Poisson and Exponential Distributions

- The Poisson distribution provides an appropriate description of the number of occurrences per interval

- The exponential distribution provides an appropriate description of the length of the interval between occurrences

Relationship between Exponential and Poisson

- If we know that there are on average 10 customers visiting a store within 2-hour interval, then the average time between customers' arrival is: 120/10=12 minutes.
- Therefore, the time interval between customer visits follows an exponential distribution with mean=12 minutes.

Expected Value

The expected value of an exponential random variable X is:

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

The proof

It can be derived as follows:

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty x f_X(x)\,dx \\
&= \int_0^\infty x\lambda \exp(-\lambda x)\,dx \\
&= [-x\exp(-\lambda x)]_0^\infty + \int_0^\infty \exp(-\lambda x)\,dx \qquad \text{(integrating by parts)} \\
&= (0-0) + \left[-\frac{1}{\lambda}\exp(-\lambda x)\right]_0^\infty \\
&= 0 + \left(0 + \frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

Variance

The variance of an exponential random variable X is:

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

The proof

It can be derived thanks to the usual variance formula ($\text{Var}[X] = \text{E}\left[X^2\right] - \text{E}[X]^2$):

$$\text{E}\left[X^2\right] = \int_0^\infty x^2 \lambda \exp(-\lambda x) dx$$

$$= \left[-x^2 \exp(-\lambda x)\right]_0^\infty + \int_0^\infty 2x \exp(-\lambda x) dx \qquad \text{(integrating by parts)}$$

$$= (0-0) + \left[-\frac{2}{\lambda} x \exp(-\lambda x)\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty \exp(-\lambda x) dx \qquad \text{(integrating by parts again)}$$

$$= (0-0) + \frac{2}{\lambda}\left[-\frac{1}{\lambda}\exp(-\lambda x)\right]_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\text{E}[X]^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\text{Var}[X] = \text{E}\left[X^2\right] - \text{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moment generating function

The moment generating function of an exponential random variable X is defined for any $t < \lambda$:

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

The proof

Using the definition of moment generating function:

$$\begin{aligned} M_X(t) &= \mathrm{E}[\exp(tX)] \\ &= \int_{-\infty}^{\infty} \exp(tx) f_X(x)dx \\ &= \int_0^{\infty} \exp(tx)\lambda \exp(-\lambda x)dx \\ &= \lambda \int_0^{\infty} \exp((t-\lambda)x)dx \\ &= \lambda \left[\frac{1}{t-\lambda} \exp((t-\lambda)x) \right]_0^{\infty} \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

Of course, the above integrals converge only if $(t-\lambda) < 0$, i.e. only if $t < \lambda$. Therefore, the moment generating function of an exponential random variable exists for all $t < \lambda$.