Solutions of Exercises Sheet #5

SOLUTION 1:

a)

$$F(x) = \int_0^x \frac{2}{\beta^2} x e^{\left(-\left(x/\beta\right)^2\right)} dx$$

$$u = -\left(x/\beta\right)^2, \; du = \frac{-2}{\beta^2} x \, dx$$

$$F(u) = -\int e^u \, du = -e^u \Rightarrow F(x) = -e^{-x^2/\beta^2} \Big|_0^x = -e^{-x^2/\beta^2} + 1$$

The inverse of the CDF is:

$$F(x) = -e^{-x^2/\beta^2} + 1$$

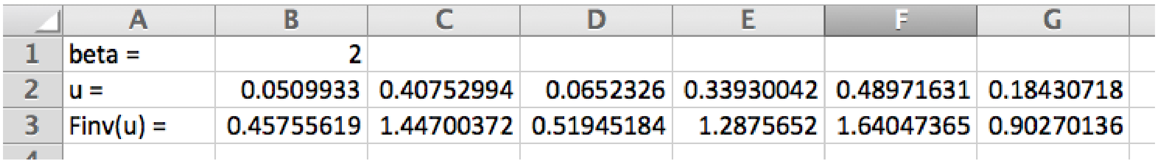
$$U = -e^{-x^2/\beta^2} + 1$$

$$\ln(1-U) = \frac{-x^2}{\beta^2} \Rightarrow -x^2 = \beta^2 \ln(1-U) \Rightarrow x = \sqrt{-\beta^2 \ln(1-U)}$$

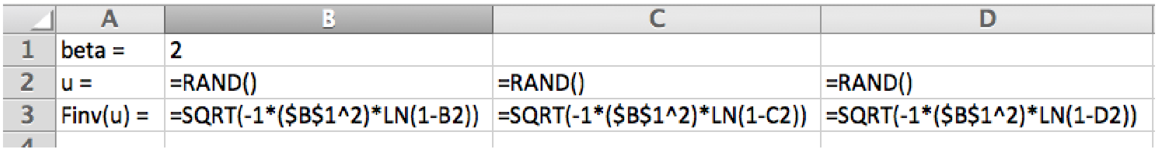
b)

Using the inverse CDF from above, with β = 2.0, and the uniform numbers given it is yields:

u =	0.943	0.398	0.372	0.943	0.204	0.794
$F^{-1}(u)$ =	3.385087302	1.424777644	1.36413359	3.385087302	0.955313756	2.513864841



	A	B	C	D	E	F	G
1	beta =	2					
2	u =	0.0509933	0.40752994	0.0652326	0.33930042	0.48971631	0.18430718
3	Finv(u) =	0.45755619	1.44700372	0.51945184	1.2875652	1.64047365	0.90270136



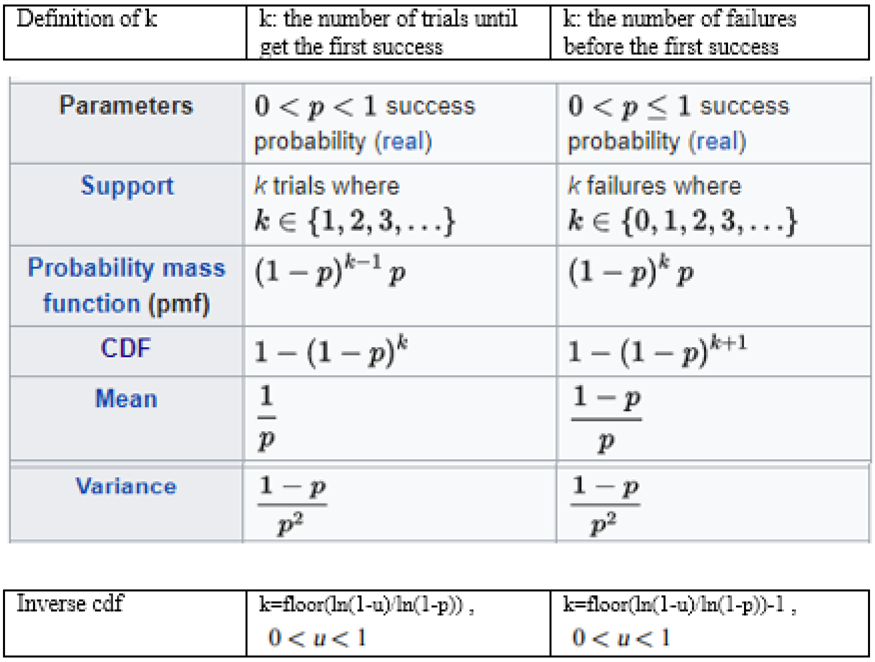
	A	B	C	D
1	beta =	2		
2	u =	=RAND()	=RAND()	=RAND()
3	Finv(u) =	=SQRT(-1*(B1^2)*LN(1-B2))	=SQRT(-1*(B1^2)*LN(1-C2))	=SQRT(-1*(B1^2)*LN(1-D2))

SOLUTION 2:

a)

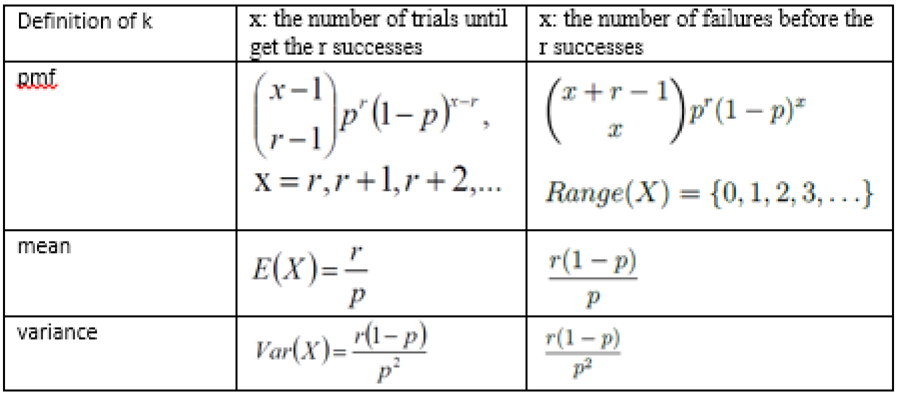
Negative Binomial = Σ *iid* Geometric variables

Geometric



Definition of k	k: the number of trials until get the first success	k: the number of failures before the first success
Parameters	$0 < p < 1$ success probability (real)	$0 < p \leq 1$ success probability (real)
Support	k trials where $k \in \{1, 2, 3, \dots\}$	k failures where $k \in \{0, 1, 2, 3, \dots\}$
Probability mass function (pmf)	$(1-p)^{k-1}\,p$	$(1-p)^k\,p$
CDF	$1-(1-p)^k$	$1-(1-p)^{k+1}$
Mean	$\frac{1}{p}$	$\frac{1-p}{p}$
Variance	$\frac{1-p}{p^2}$	$\frac{1-p}{p^2}$
Inverse cdf	k=floor(ln(1-u)/ln(1-p)) , $0 < u < 1$	k=floor(ln(1-u)/ln(1-p))-1 , $0 < u < 1$

Negative Binomial

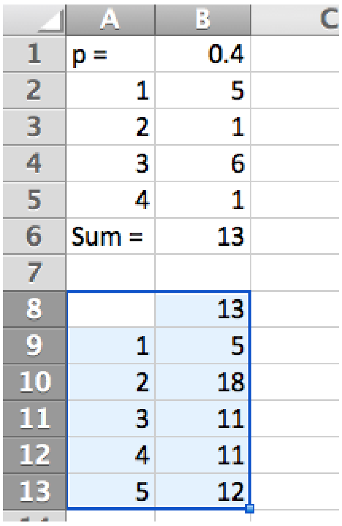


Definition of k	x: the number of trials until get the r successes	x: the number of failures before the r successes
pmf	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$, $X = r, r+1, r+2, \dots$	$\binom{x+r-1}{x}p^r(1-p)^x$ $Range(X) = \{0, 1, 2, 3, \dots\}$
mean	$E(X) = \frac{r}{p}$	$\frac{r(1-p)}{p}$
variance	$Var(X) = \frac{r(1-p)}{p^2}$	$\frac{r(1-p)}{p^2}$

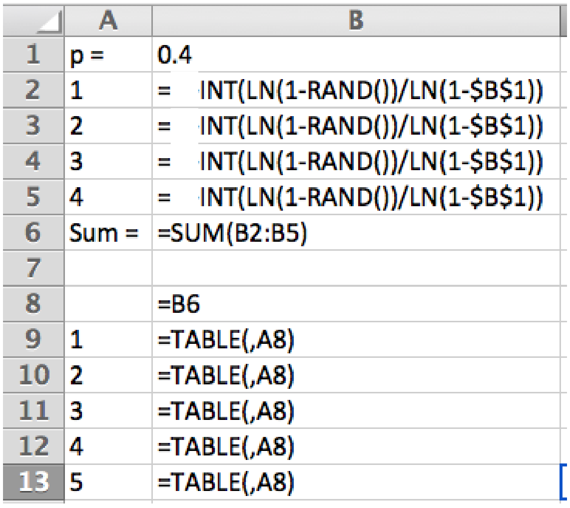
Convolution method: The negative binomial distribution (r = 4, p = 0.4) is the sum of 4 geometric random variables with (p = 0.4).

U	GEOM(p=0.4) = floor(ln(1-u)/ln(1-p))
0.943	5
0.498	1
0.102	0
0.398	0

Answer: 6 trials



	A	B	C
1	p =	0.4	
2	1	5	
3	2	1	
4	3	6	
5	4	1	
6	Sum =	13	
7			
8		13	
9	1	5	
10	2	18	
11	3	11	
12	4	11	
13	5	12	



	A	B
1	p =	0.4
2	1	= INT(LN(1-RAND())/LN(1-B1))
3	2	= INT(LN(1-RAND())/LN(1-B1))
4	3	= INT(LN(1-RAND())/LN(1-B1))
5	4	= INT(LN(1-RAND())/LN(1-B1))
6	Sum =	=SUM(B2:B5)
7		
8		=B6
9	1	=TABLE(,A8)
10	2	=TABLE(,A8)
11	3	=TABLE(,A8)
12	4	=TABLE(,A8)
13	5	=TABLE(,A8)

b)

<u>Bernoulli (*p*)</u>

X ~ Bernoulli(p)

Pr{X =1} = p and Pr{X = 0} = 1 − p

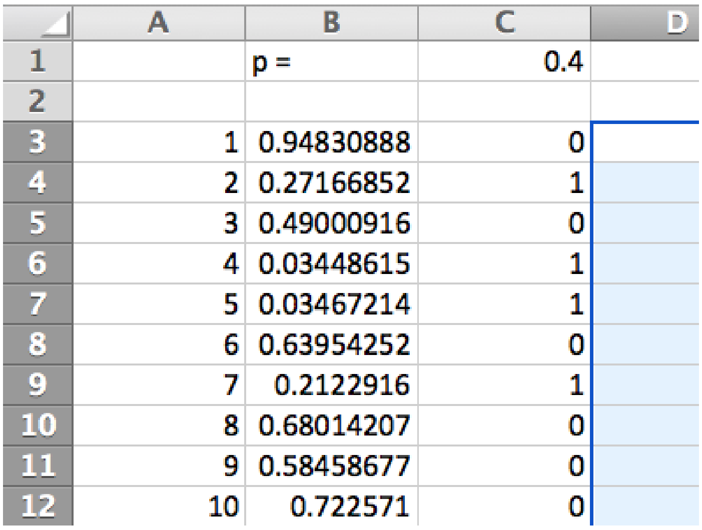
For u~ U[0,1]

$$F(u)^{-1} = \begin{cases} 1 & ;\ 0 \le u \le p \\ 0 & ;\ p < u \le 1 \end{cases}$$

Bernoulli trials: Generate Bernoulli trials ($p = 0.4$)until you get 4 successes

	U	Bernoulli trial
1	0.943	0
2	0.498	0
3	0.102	1
4	0.398	1
5	0.528	0
6	0.057	1
7	0.372	1

Answer: 7 trials



	A	B	C	D
1		p =	0.4	
2				
3	1	0.94830888	0	
4	2	0.27166852	1	
5	3	0.49000916	0	
6	4	0.03448615	1	
7	5	0.03467214	1	
8	6	0.63954252	0	
9	7	0.2122916	1	
10	8	0.68014207	0	
11	9	0.58458677	0	
12	10	0.722571	0	

SOLUTION 3:

This is a mixture distribution. Let F_1 represent the lognormal distribution with $\omega_1 = 0.3$. Let F_2 represent the uniform distribution with $\omega_2 = 0.7$.

```
Generate u ~ U(0,1)
Generate v ~ U(0,1)
     If u<= 0.3  then
        x = a + (b-a)u
     Else
        x= e^(NORM.INV(v, $\mu$, $\sigma^2$))
     End if
 Return x
```

If $X \sim \mathcal{N}(\mu, \sigma^2)$ is a normal distribution, then $\exp(X) \sim \mathbf{Lognormal}(\mu, \sigma^2)$

By Excel:

Generate Y ~ N(μ, σ^2) via NORM.INV(v, μ, σ), then X = EXP(Y) will be lognormal, where v will be the U(0,1) and by let

$$m = E[X]$$
$$v = V[X]$$

Then,

$$\mu = ln\left(\frac{m}{\sqrt{1 + \frac{v}{m^2}}}\right)$$

$$\sigma^2 = ln\left(1 + \frac{v}{m^2}\right)$$

Using U1 = 0.943 to pick the distribution implies, X ~ U(10,20) because 0.943 > 0.3

X = a + (b-a)U2 = 10 + 10*0.398 = 13.98

Using U3 = 0.372 to pick the distribution implies, X ~ U(10,20) because 0.372 > 0.3

X = a + (b-a)U4 = 10 + 10*0.943 = 19.43

We “got lucky” and did not have to generate from the lognormal distribution.

SOLUTION 4:

This is a mixture distribution. Let F_1 represent the U(20,25) distribution with $\omega_1 = 0.25$. Let F_2 represent the Weibull distribution($\alpha = 2, \beta = 4.5$) with $\omega_2 = 0.75$.

```
Generate u ~ U(0,1)
Generate v ~ U(0,1)
     If u<= 0.25  then
        x = a + (b-a)u
     Else
        x= $\beta[-ln(1-v)]^{\frac{1}{\alpha}}$
     End if
Return x
```

0.943	0.398	0.372	0.943	0.204	0.794
0.498	0.528	0.272	0.899	0.294	0.156
0.102	0.057	0.409	0.398	0.400	0.997

Using U1 = 0.943 to pick the distribution implies, X ~ Weibull because 0.943 > 0.25

Using U2 = 0.398

X = 4.5[-ln(1-0.398)]^(1/2) = 3.2057

Using U3 = 0.372 to pick the distribution implies, X ~ Weibull because 0.372 > 0.25

Using U4 = 0.943

X = 4.5[-ln(1-0.943)]^(1/2) = 7.616

SOLUTION 5:

Chi-squared Variable = Σ *iid* Squared normal variables.

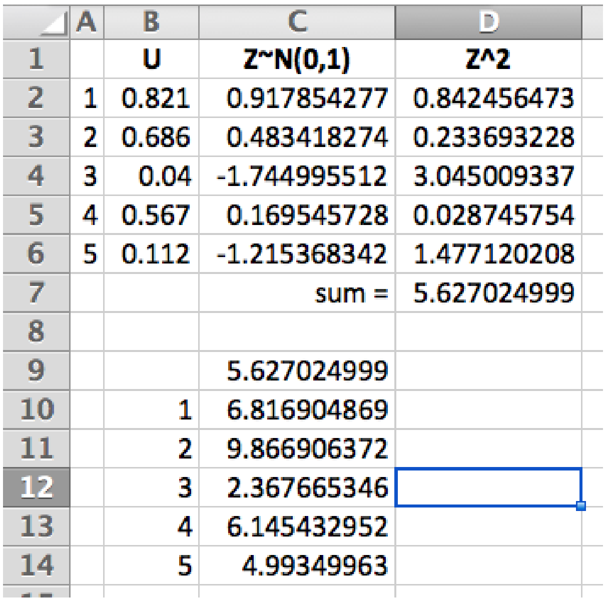
i.e.

If X_1, X_2, …, X_n are independent standard normal random variables, then the sum of their squares has the chi-squared distribution with n degrees of freedom

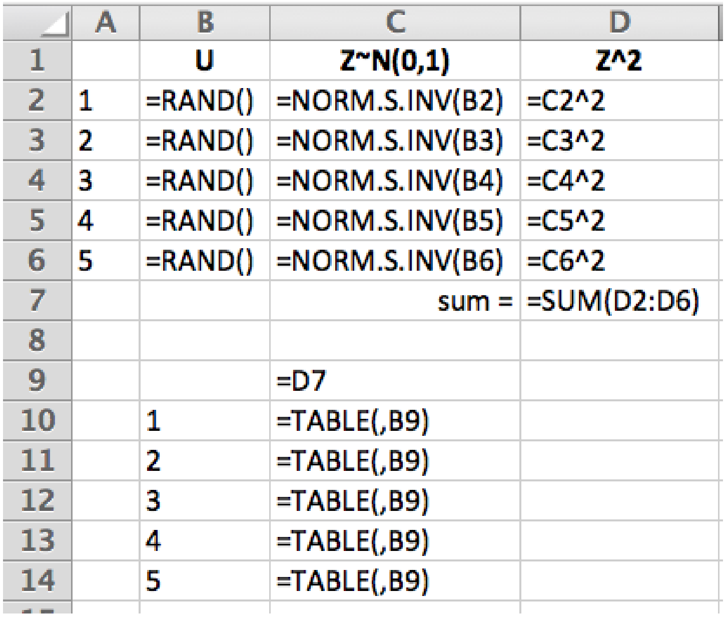
$$X_1^2 + \cdots + X_n^2 \sim \chi_n^2$$

Use Z = NORM.S.INV(U) where U is read from the table. Do this for 5 PRN's and, square and sum the values. Students could also use the z-table or by Excel.

	U	**Z~N(0,1)**	**Z^2**	
1	0.943	1.580466818	2.497875364	
2	0.398	-0.258527277	0.066836353	
3	0.372	-0.326560927	0.106642039	
4	0.943	1.580466818	2.497875364	
5	0.204	-0.827418321	0.684621077	
		sum =	5.853850198	Y1
1	0.794	0.820379146	0.673021943	
2	0.498	-0.005013278	2.5133E-05	
3	0.528	0.070243314	0.004934123	
4	0.272	-0.606775364	0.368176342	
5	0.899	1.275874179	1.627854921	
		sum =	2.674012462	Y2



	A	B	C	D
1		**U**	**Z~N(0,1)**	**Z^2**
2	1	0.821	0.917854277	0.842456473
3	2	0.686	0.483418274	0.233693228
4	3	0.04	-1.744995512	3.045009337
5	4	0.567	0.169545728	0.028745754
6	5	0.112	-1.215368342	1.477120208
7			sum =	5.627024999
8				
9			5.627024999	
10		1	6.816904869	
11		2	9.866906372	
12		3	2.367665346	
13		4	6.145432952	
14		5	4.99349963	



	A	B	C	D
1		**U**	**Z~N(0,1)**	**Z^2**
2	1	=RAND()	=NORM.S.INV(B2)	=C2^2
3	2	=RAND()	=NORM.S.INV(B3)	=C3^2
4	3	=RAND()	=NORM.S.INV(B4)	=C4^2
5	4	=RAND()	=NORM.S.INV(B5)	=C5^2
6	5	=RAND()	=NORM.S.INV(B6)	=C6^2
7			sum =	=SUM(D2:D6)
8				
9			=D7	
10		1	=TABLE(,B9)	
11		2	=TABLE(,B9)	
12		3	=TABLE(,B9)	
13		4	=TABLE(,B9)	
14		5	=TABLE(,B9)	

SOLUTION 6:

(a) acceptance/rejection

(b) majorizing

c) acceptance probability

SOLUTION 7:

a)

Choose $g(x) = 3/2$. Integrating over [-1, 1] yields $c = 3$. Thus, $w(x) = ½$ over [-1,1]

Algorithm

```
Generate $u \sim U(0,1)$
 $x=2u-1$
 Generate $v \sim U(0,1)$
        If $f(x)/g(x) >= v$ then
            $x \sim f(x)$
        Else
            $x \nsim f(x)$
         End if
   Return x
```

Inverse cdf of w(x)
or
we can see direct that
$x = a + (b-a)* u = -1 + (1 - -1)u = 2*u -1$

Accept

$f(x)/g(x) < v$

Reject

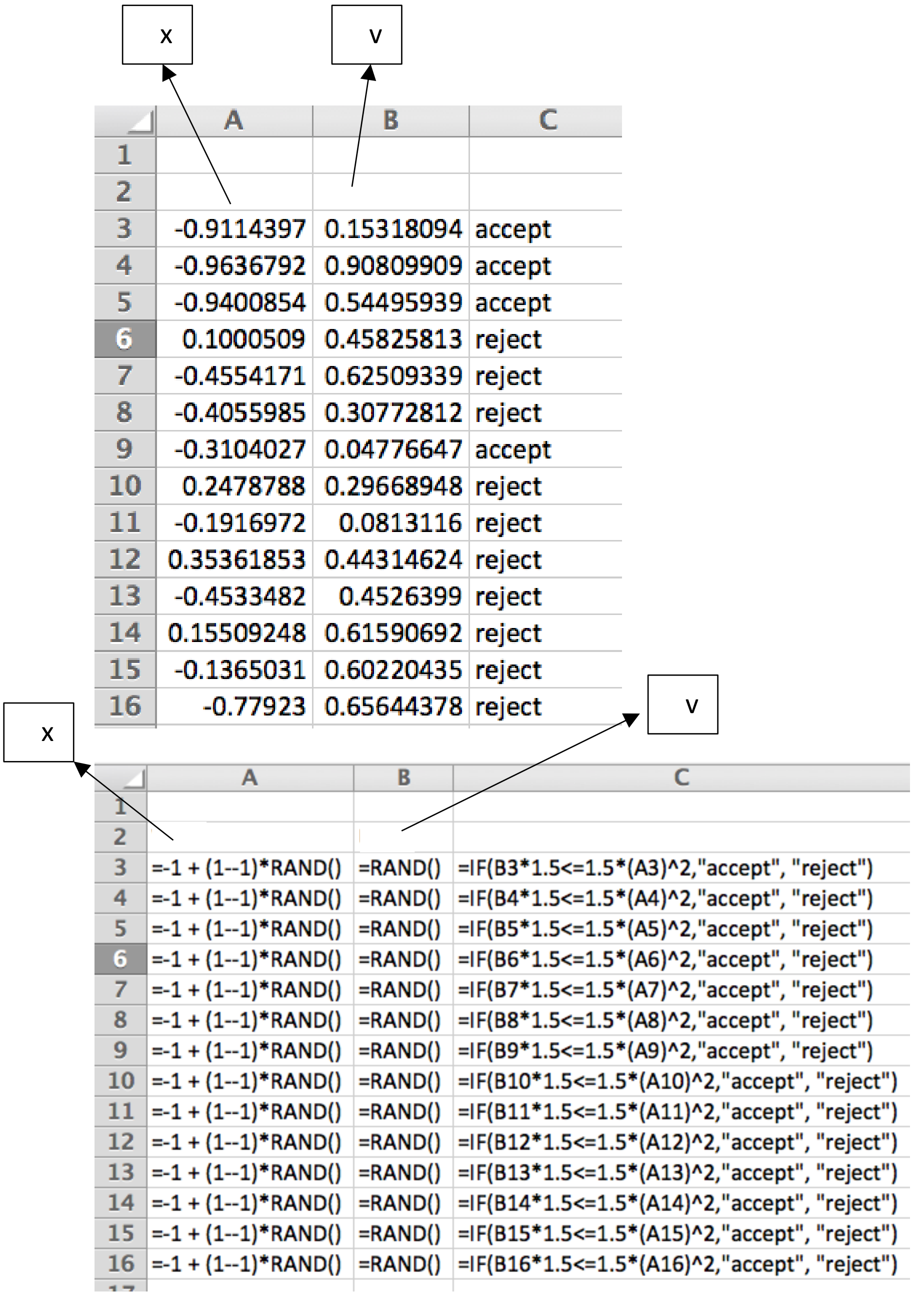
where $f(x)/g(x)= x^2$

b)

$u = 0.943 \rightarrow x = 2*0.943 -1 = 0.886$
$v = 0.398$
as $f(0.886)/g(0.886)=.785 > v$, therefore accept $x = 0.886$

$u = 0.372 \rightarrow x= 2*0.372 -1 = -0.256$
$v = 0.943$
as $f(-0.256)/g(-0.256)=.066 < v$, therefore reject $x = 0.886$

Continue in this manner until you get the 2nd acceptance.



	A	B	C
1			
2			
3	-0.9114397	0.15318094	accept
4	-0.9636792	0.90809909	accept
5	-0.9400854	0.54495939	accept
6	0.1000509	0.45825813	reject
7	-0.4554171	0.62509339	reject
8	-0.4055985	0.30772812	reject
9	-0.3104027	0.04776647	accept
10	0.2478788	0.29668948	reject
11	-0.1916972	0.0813116	reject
12	0.35361853	0.44314624	reject
13	-0.4533482	0.4526399	reject
14	0.15509248	0.61590692	reject
15	-0.1365031	0.60220435	reject
16	-0.77923	0.65644378	reject

	A	B	C
1			
2			
3	=-1 + (1--1)*RAND()	=RAND()	=IF(B3*1.5<=1.5*(A3)^2,"accept", "reject")
4	=-1 + (1--1)*RAND()	=RAND()	=IF(B4*1.5<=1.5*(A4)^2,"accept", "reject")
5	=-1 + (1--1)*RAND()	=RAND()	=IF(B5*1.5<=1.5*(A5)^2,"accept", "reject")
6	=-1 + (1--1)*RAND()	=RAND()	=IF(B6*1.5<=1.5*(A6)^2,"accept", "reject")
7	=-1 + (1--1)*RAND()	=RAND()	=IF(B7*1.5<=1.5*(A7)^2,"accept", "reject")
8	=-1 + (1--1)*RAND()	=RAND()	=IF(B8*1.5<=1.5*(A8)^2,"accept", "reject")
9	=-1 + (1--1)*RAND()	=RAND()	=IF(B9*1.5<=1.5*(A9)^2,"accept", "reject")
10	=-1 + (1--1)*RAND()	=RAND()	=IF(B10*1.5<=1.5*(A10)^2,"accept", "reject")
11	=-1 + (1--1)*RAND()	=RAND()	=IF(B11*1.5<=1.5*(A11)^2,"accept", "reject")
12	=-1 + (1--1)*RAND()	=RAND()	=IF(B12*1.5<=1.5*(A12)^2,"accept", "reject")
13	=-1 + (1--1)*RAND()	=RAND()	=IF(B13*1.5<=1.5*(A13)^2,"accept", "reject")
14	=-1 + (1--1)*RAND()	=RAND()	=IF(B14*1.5<=1.5*(A14)^2,"accept", "reject")
15	=-1 + (1--1)*RAND()	=RAND()	=IF(B15*1.5<=1.5*(A15)^2,"accept", "reject")
16	=-1 + (1--1)*RAND()	=RAND()	=IF(B16*1.5<=1.5*(A16)^2,"accept", "reject")

SOLUTION 8:

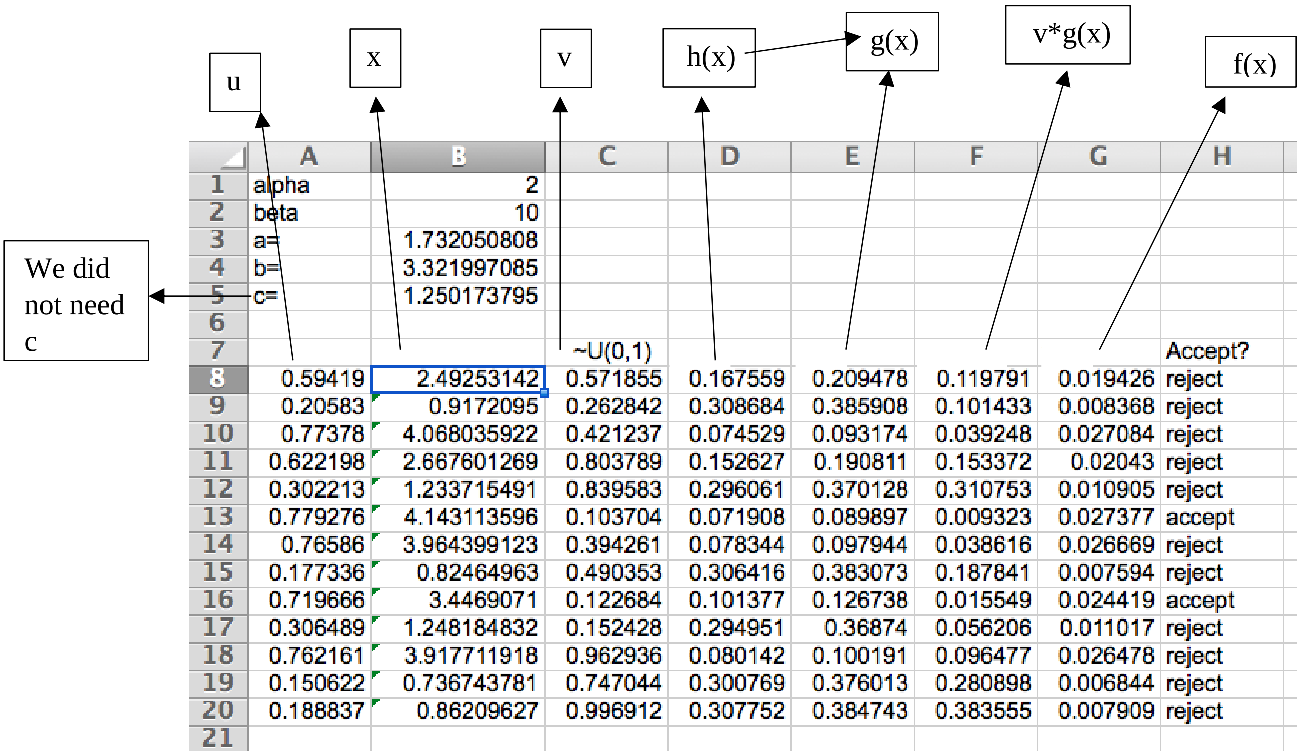
a)

$$\text{w(x)} = h(x) = ab\frac{x^{a-1}}{\left(b+x^{a}\right)^{2}} \qquad for\ x>0$$

$$\rightarrow \text{cdf} = \frac{x^{a}}{\left(b+x^{a}\right)} \qquad for\ x>0$$

$$\rightarrow \text{inverce of cdf} = x = \left(\frac{bu}{1-u}\right)^{1/a}$$

b)



	A	B	C	D	E	F	G	H
1	alpha	2						
2	beta	10						
3	a=	1.732050808						
4	b=	3.321997085						
5	c=	1.250173795						
6								
7			~U(0,1)					Accept?
8	0.59419	2.49253142	0.571855	0.167559	0.209478	0.119791	0.019426	reject
9	0.20583	0.9172095	0.262842	0.308684	0.385908	0.101433	0.008368	reject
10	0.77378	4.068035922	0.421237	0.074529	0.093174	0.039248	0.027084	reject
11	0.622198	2.667601269	0.803789	0.152627	0.190811	0.153372	0.02043	reject
12	0.302213	1.233715491	0.839583	0.296061	0.370128	0.310753	0.010905	reject
13	0.779276	4.143113596	0.103704	0.071908	0.089897	0.009323	0.027377	accept
14	0.76586	3.964399123	0.394261	0.078344	0.097944	0.038616	0.026669	reject
15	0.177336	0.82464963	0.490353	0.306416	0.383073	0.187841	0.007594	reject
16	0.719666	3.4469071	0.122684	0.101377	0.126738	0.015549	0.024419	accept
17	0.306489	1.248184832	0.152428	0.294951	0.36874	0.056206	0.011017	reject
18	0.762161	3.917711918	0.962936	0.080142	0.100191	0.096477	0.026478	reject
19	0.150622	0.736743781	0.747044	0.300769	0.376013	0.280898	0.006844	reject
20	0.188837	0.86209627	0.996912	0.307752	0.384743	0.383555	0.007909	reject
21								

SOLUTION 9:

This question demonstrates the splitting property of a Poisson distribution. Each machine experience a Poisson process with mean $\lambda \times p_i$. Thus, the distribution of the inter-arrival times to each drill press will be exponential with mean $1/(\lambda \times p_i)$

Because of the splitting rule for Poisson processes, the drill presses each see arrivals according to the following three Poisson processes:

$$\lambda_1 = \lambda p_1 = 12 * 0.25 = 3$$

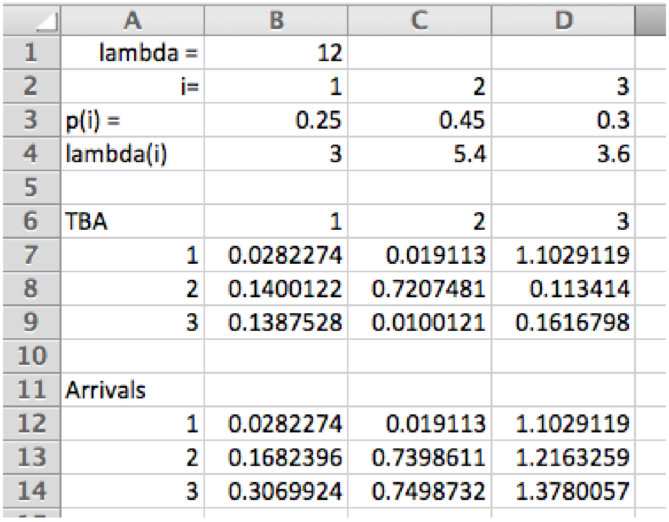
$$\lambda_2 = \lambda p_2 = 12 * 0.45 = 5.4$$

$$\lambda_3 = \lambda p_3 = 12 * 0.3 = 3.6$$

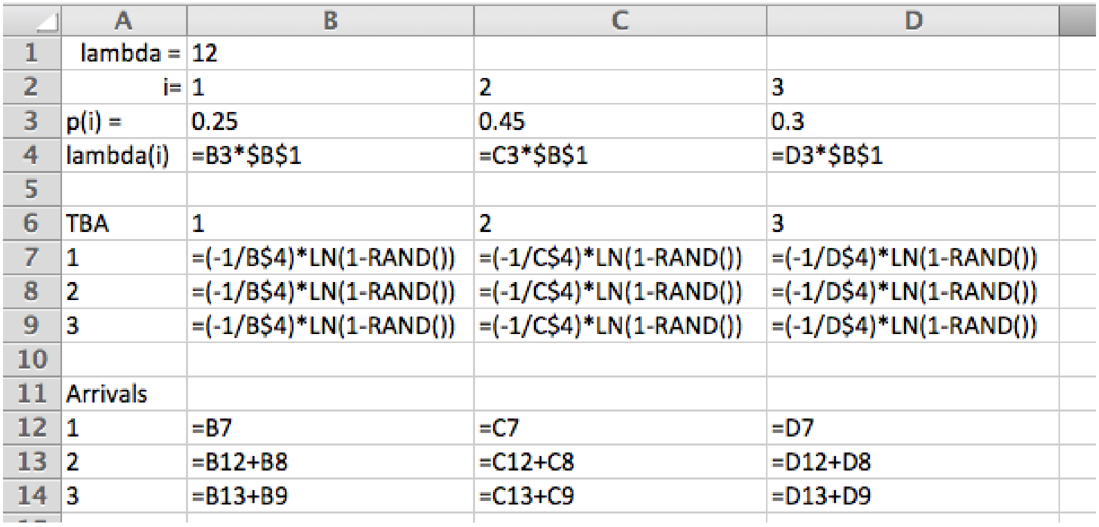
Since the time between arrivals will be exponential, we have the following first arrival time to each drill press:

X1 = -(1/3)ln(1-0.943) = 0.9549
X2 = -(1/5.4)ln(1-0.398) = 0.09398
X3 = -(1/3.6)ln(1-0.372) = 0.12923

Generate from 3 different exponential distributions using these rates.



	A	B	C	D
1	lambda =	12		
2	i=	1	2	3
3	p(i) =	0.25	0.45	0.3
4	lambda(i)	3	5.4	3.6
5				
6	TBA	1	2	3
7	1	0.0282274	0.019113	1.1029119
8	2	0.1400122	0.7207481	0.113414
9	3	0.1387528	0.0100121	0.1616798
10				
11	Arrivals			
12	1	0.0282274	0.019113	1.1029119
13	2	0.1682396	0.7398611	1.2163259
14	3	0.3069924	0.7498732	1.3780057



	A	B	C	D
1	lambda =	12		
2	i=	1	2	3
3	p(i) =	0.25	0.45	0.3
4	lambda(i)	=B3*B1	=C3*B1	=D3*B1
5				
6	TBA	1	2	3
7	1	=(-1/B$4)*LN(1-RAND())	=(-1/C$4)*LN(1-RAND())	=(-1/D$4)*LN(1-RAND())
8	2	=(-1/B$4)*LN(1-RAND())	=(-1/C$4)*LN(1-RAND())	=(-1/D$4)*LN(1-RAND())
9	3	=(-1/B$4)*LN(1-RAND())	=(-1/C$4)*LN(1-RAND())	=(-1/D$4)*LN(1-RAND())
10				
11	Arrivals			
12	1	=B7	=C7	=D7
13	2	=B12+B8	=C12+C8	=D12+D8
14	3	=B13+B9	=C13+C9	=D13+D9

Alternative solution procedure:

Generate inter-arrival times by using λ=12. At each arrival, determine which drill press sees the arrival by using the PMF (0.25, 0.45, 0.3) to pick the drill press. Continue generating until you get the first arrival at each drill press.