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Solution of Homework n°6
Math 580 (Theory measure)

Exercise n°1

As $fg \geq 1$ then $\sqrt{f}\,\sqrt{g} \geq 1$

We apply now Cauchy-Shwartz Inequality:

$$\int_X f(x)g(x)\,d\mu(x) \leq \left(\int_X f^2\,d\mu(x)\right)^{1/2}\left(\int_X g^2\,d\mu(x)\right)^{1/2}$$

We get

$$\int_X 1\,d\mu(x) \leq \int_X \sqrt{fg}\,d\mu(x) \leq \left(\int_X f(x)\,d\mu(x)\right)^{1/2}\left(\int_X g(x)\,d\mu(x)\right)^{1/2}$$

So $$\mu(X)^2 \leq \left(\int_X f(x)\,d\mu(x)\right)\left(\int_X g(x)\,d\mu(x)\right)$$

As $\mu(X)=1$ ($(X,\mathcal{A},\mu)$ is a probabity space).

then $$\left(\int_X f(x)\,d\mu(x)\right)\cdot\left(\int_X g(x)\,d\mu(x)\right) \geq 1$$

Exercise n°2

We apply Hölder's inequality with $p=3$,

$$\int_X |f_n(x)|\frac{1}{\sqrt{x}}\,d\lambda(x) \leq \left(\int_X |f_n(x)|^3\,d\lambda(x)\right)^{1/3}\left(\int_X \left(\frac{1}{\sqrt{x}}\right)^{3/2}d\lambda(x)\right)^{2/3}$$

$$\int_X \left(\frac{1}{\sqrt{x}}\right)^{3/2}d\lambda(x) = \int_0^1 x^{-3/4}\,dx = 4\left[x^{1/4}\right]_0^1 = 4$$

$$0 \leq \lim_{n\to\infty}\int_X \left|\frac{f_n(x)}{\sqrt{x}}\right|d\lambda(x) \leq 4^{2/3}\lim_{n\to\infty}\left(\int_X |f_n(x)|^3\,d\lambda(x)\right)^{1/3} \leq 0$$

So $$\lim_{n\to\infty}\int_X \left|\frac{f_n(x)}{\sqrt{x}}\right|d\lambda(x) = 0 \Rightarrow \lim_{n\to\infty}\int_X \frac{f_n(x)}{\sqrt{x}}\,d\lambda(x) = 0.$$

Exercise n°3

① a) let $p \geq 1$, we consider $\varphi(x) = x^p$, $\varphi''(x) = p(p-1)x^{p-2} \geq 0$ for $x > 0$.

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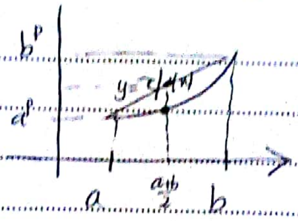
For $a > b > 0$. As ψ convex,



We have: $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p+b^p}{2}$

Now we take $a=|f|$ and $b=|f_n|$

$$\frac{|f-f_n|^p}{2^p} \leq \frac{(|f|+|f_n|)^p}{2^p} \leq \frac{|f|^p+|f_n|^p}{2}, \ \forall n$$

So

$$\varphi_n(x) = 2^{p-1}\left(|f|^p + |f_n|^p - |f-f_n|^p\right) \geq 0, \ \forall x \in X$$

b) Using Fatou's lemma for φ_n:

$\varphi_n \geq 0$

φ_n measurable

$$0 \leq \int_X \lim_{n\to\infty} \varphi_n(x)\, d\mu(x) \leq \underline{\lim}_{n\to\infty} \int_X \varphi_n(x)\, d\mu(x)$$

$$0 \leq \int_X 2^p |f(x)|^p d\mu(x) \leq 2^{p-1} \underline{\lim}_{n\to\infty} \int_X \left(\left(|f(x)|^p + |f_n(x)|^p\right) - |f(x)-f_n(x)|^p\right) d\mu$$

$$\leq 2^{p-1}\left[\int_X |f(x)|^p d\mu(x) + \underline{\lim} \int_X |f_n(x)|^p d\mu - \underline{\lim} \int_X |f-f_n|\right.$$

As f_n converges to f a.e., it means $\lim_{n\to\infty} |f_n - f| = 0$

and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$, it means $\lim_{n\to\infty} \int_X |f_n|^p d\mu = \int_X |f|^p d\mu$

It follows that:

$$\int_X 2^p |f(x)|^p d\mu(x) \leq 2^p \int_X |f(x)|^p d\mu(x) + 2^{p-1} \underbrace{\underline{\lim} \int_X -|f-f_n|^p d\mu}_{= -\overline{\lim} \int_X |f-f_n|^p d\mu}$$

Then $0 \geq \overline{\lim}\left(\int_X |f-f_n|^p d\mu(x)\right)$.

$$\boxed{\lim\sup(x_n) = -\lim\inf(-x_n)}$$

$$\overline{\lim}\left(\int_X |f-f_n|^p d\mu(x)\right) \leq 0$$

So, $\|f-f_n\|_p \xrightarrow[n\to\infty]{} 0$. It means that: $\lim_{n\to\infty} f_n = f$ in L^p

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② We take $f_n(x) = \chi_{[n,n+1]}(x)$,

- $f_n \xrightarrow[n\to\infty]{} 0$ a.e
- $f_n \in L^1(\mathbb{R})$ because $\int_{\mathbb{R}} \chi_{[n,n+1]}(x)\, d\lambda(x) < \infty$, $\forall n \geq 0$.
- But $\int_X |\chi_{[n,n+1]}(x) - 0|\, d\lambda(x) = \int_n^{n+1} dx = 1$, For all $n \geq 0$.

It means that $f_n \cancel{\xrightarrow[n\to\infty]{}} 0$ in $L^1(\mathbb{R})$.

Exercise n°4 :

① $\lim_{n\to\infty} \int_X |f_n - f|\, d\mu \stackrel{?}{=} 0$

With the hypotheses $f_n \xrightarrow[n\to\infty]{} f$ a.e and $\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$

We can not apply Dominate Convergence Thm for $(|f_n - f|)_{n\geq 1}$.

We consider $g_n = \min(f_n, f)$

We have

$$\lim_{n\to\infty} g_n = f \quad \text{a.e} \quad \text{and} \quad g_n \text{ is dominated by } f \in L^1. \; (g_n \leq f)$$

We apply now Dominate Convergence Thm for g_n;

we get $\lim_{n\to\infty} \int_X g_n\, d\mu = \int_X f\, d\mu$

But $|f_n - f| = f_n + f - 2g_n$. So

$$\int_X |f_n - f|\, d\mu = \int_X f_n\, d\mu + \int_X f\, d\mu - 2\int_X g_n\, d\mu$$

It follows that

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0 \iff f_n \xrightarrow[n\to\infty]{} f \text{ in } L^1.$$

② $f_n = n\,\chi_{(0,1/n)}(x) - n\,\chi_{(-1/n,0)}(x)$

For all $x \neq 0$, if n is big, $|x| > 1/n$ and $f_n(x) = 0$

and $f_n(0)=0$. Then (f_n) converges pointwise to 0.

It follows that $\forall n\geq 1$, $\int_{\mathbb{R}} f_n(x)\,d\lambda(x)=0$

So $\lim\limits_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,d\lambda(x)=0$

③ let $p\geq 1$, As $\int_{\mathbb{R}} |f_n|^p d\lambda = 2n^{p-1} \underset{n\to\infty}{\not\longrightarrow} 0$

So (f_n) does not converge to 0 in L^p.

Exercise n°5 :

① let $0<p<1$. We have $\frac{1}{(1/p)}+\frac{1}{(1/1-p)}=1$.

Now we use Hölder's inequality

$$\int_X f^p\,d\mu = \int_X f^p\cdot\chi_{\{f>0\}}\,d\mu$$

$$\leq \left(\int_X (f^p)^{1/p} d\mu\right)^{1/1/p}\cdot\left(\int_X \chi_{\{f>0\}}^{1/1-p}\,d\mu\right)^{1/1/1-p}$$

$$\leq \left(\int_X f\,d\mu\right)^{p}\cdot\left(\int_X \chi_{\{f>0\}}\,d\mu\right)^{1-p}$$

If $\mu(\{f>0\})=\int_X \chi_{\{f>0\}}\,d\mu<1$, we get,

$$\|f\|_p^p \leq \left(\int_X f\,d\mu\right)^p\cdot\left(\mu(\{f>0\})\right)^{1-p}$$

$$\Rightarrow \|f\|_p \leq \left(\int_X f\,d\mu\right)\cdot\left(\mu(\{f>0\})\right)^{\frac{1-p}{p}} \xrightarrow[p\to 0^+]{} 0$$

$$\mu(\{f>0\})^{\frac{1-p}{p}} = e^{\left(\frac{1-p}{p}\right)\ln(\mu(\{f>0\}))} \xrightarrow[p\to 0^+]{} 0 = e^{-\infty}$$

because $\mu(\{f>0\})<1$.

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(2) $\lim_{p\to 0^+} \int_X f^p \, d\mu \overset{?}{=} \mu(\{f>0\})$

If $0<p<1$ then $|f^p| \leq 1+|f| = 1+f$. ($f\geq 0$)

But $(1+f)$ is μ-integrable because f is μ-integrable.

In other hand, $\lim_{p\to 0^+} f^p = \chi_{\{f>0\}}$

Now we apply Dominate Convergence Thm for $(f^p)_p$

$$\lim_{p\to 0^+} \int_X f^p d\mu = \int_X \chi_{\{f>0\}} d\mu = \mu(\{f>0\}).$$

(3) let $p\in(0,1)$. If $\boxed{x\in(0,1)}$, we apply Mean value Thm for the function $p \overset{\varphi}{\longmapsto} x^p$ on $(0,p)$.

$$\frac{x^p - x^0}{p} = \ln x \, e^{\xi \ln x} \quad \text{where } 0<\xi<p$$

As $0<x<1$ then $\ln x<0$

$\Rightarrow \xi \ln x < 0$

$e^{\xi \ln x} < e^0 = 1$

So $$\frac{|x^p-1|}{p} \leq |\ln x| \quad \text{for all } x\in(0,1)$$

Now if $\boxed{x\geq 1}$. We apply Mean value Thm for the function $x \overset{\psi}{\longmapsto} x^p$ on $[1,x]$. we obtain

$$\frac{x^p - 1^p}{x-1} = p\, z^{p-1} \quad \text{where } 1<z<x \text{ and } \forall p\in(0,1)$$

$$\frac{x^p-1}{p} = (x-1)\, z^{p-1}$$

So $$\frac{|x^p-1|}{p} \leq x$$

* We conclude that: $\forall x\in(0,\infty)$ and $\forall p\in(0,1)$, we have

$$\frac{|x^p-1|}{p} \leq x + |\ln x|$$

(4) We consider $\left(\frac{f^p-1}{p}\right)_p$ where $p\in(0,1)$

- $\frac{f^p-1}{p} \xrightarrow[p\to 0^+]{} \ln f$ pointwise.

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From question ③, we have:

$$\left|\frac{f^p-1}{p}\right| \leq |f| + |\ln f|$$

and by hypothesis f and $\ln f$ are μ-integrable, we apply Dominate Convergence Thm, we obtain

$$\lim_{p\to 0^+} \int_X \left(\frac{f^p-1}{p}\right) d\mu = \int_X \ln f \, d\mu$$

⑤ As $f>0$ on X then $\{f>0\}=X$. From question ②, we get: $\lim\limits_{p\to 0^+} \int_X f^p d\mu = \mu(X) = 1$

In other hand for $p\in(0,1)$, $\|f\|_p = \left(\int_X f^p d\mu\right)^{1/p} = \exp\left(\frac{1}{p}\ln\left(\int_X f^p d\mu\right)\right)$

As $\ln x \sim x-1$ when $x\to 1$

When $p\to 0^+$, $\frac{1}{p}\ln\left(\int_X f^p d\mu\right) \sim \frac{1}{p}\left(\int_X f^p d\mu - 1\right)$

$$\frac{1}{p}\left(\int_X f^p d\mu - 1\right) = \int_X \left(\frac{f^p-1}{p}\right) d\mu$$

From question ④, we obtain

$$\lim_{p\to 0^+} \|f\|_p = \exp\left(\int_X \ln f \, d\mu\right).$$

Exercise n°6

① For $x>0$ (fixed), and $p>1$. By Hölder inequality, we have:

$$\int_0^x f(t)\,dt = \int_{\mathbb{R}_+} f(t)\,\chi_{[0,x]}(t)\,dt \leq \left(\int_{\mathbb{R}_+} f^p(t)\,dt\right)^{1/p} \cdot \left(\int_{\mathbb{R}_+} \chi_{[0,x]}^{\frac{p}{p-1}}(t)\,dt\right)^{\frac{p-1}{p}}$$

$$\leq \left(\int f^p(t)\,dt\right)^{1/p} \cdot x^{\frac{p-1}{p}}$$

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As $f \in L^p$ then $\int_{\mathbb{R}_+} f^p(t)\,dt < \infty$.

So $\int_0^x f(t)\,dt$ exists. Hence $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$ is well-defined.

(2) Let $f \in C_K(\mathbb{R}_+^*, \mathbb{R}_+)$ (ie f is continuous on $(0,\infty)$ with compact support)

By integration by parts:

$$\int_0^\infty (F(x))^p\,dx = \left[x(F(x))^p\right]_0^\infty - p\int_0^\infty x(F(x))^{p-1}F'(x)\,dx$$

$u'(x) = 1 \Rightarrow u(x) = x$

$v(x) = (F(x))^p \quad v'(x) = p(F(x))^{p-1}F'(x)$

$$\lim_{x\to\infty} x(F(x))^p = \lim_{x\to\infty} x\cdot\frac{1}{x^p}\left(\int_0^x f(t)\,dt\right)^p = \lim_{x\to\infty}\frac{1}{x^{p-1}}\left(\int_0^x f(t)\,dt\right)^p = 0$$

because $p > 1$ and $f \in L^1$. We obtain:

$$\int_0^\infty (F(x))^p\,dx = -p\int_0^\infty x(F(x))^{p-1}F'(x)\,dx$$

As $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$ then $F'(x) = -\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{f(x)}{x}$.

$$\int_0^\infty (F(x))^p\,dx = -p\int_0^\infty x(F(x))^{p-1}\left[-\frac{1}{x^2}\int_0^x f(t)\,dt + \frac{f(x)}{x}\right]dx$$

$$= p\int_0^\infty (F(x))^p\,dx - p\int_0^\infty (F(x))^{p-1}f(x)\,dx$$

$$(1-p)\int_0^\infty (F(x))^p\,dx = -p\int_0^\infty (F(x))^{p-1}f(x)\,dx$$

Hence $\int_0^\infty (F(x))^p\,dx = \frac{p}{p-1}\int_0^\infty (F(x))^{p-1}f(x)\,dx$.

(3) Using Hölder inequality: $\frac{1}{p}+\frac{1}{q}=1$, $f \in C_K(\mathbb{R}_+^*, \mathbb{R}_+)$,

$$\int_0^\infty F(x)^{p-1}f(x)\,dx \le \left(\int_0^\infty f(x)^p\,dx\right)^{1/p}\cdot\left(\int_0^\infty F(x)^{q(p-1)}\,dx\right)^{1/q}.$$

As $q(p-1) = p$, $\int_0^\infty (F(x))^{p-1}f(x)\,dx \le \|f\|_p\left(\int_0^\infty (F(x))^p\,dx\right)^{1/q}$.

We deduce from ②:

$$\int_0^{\infty} (F(x))^p \, dx \leq \frac{p}{p-1} \|f\|_p \left(\int_0^{\infty} (F(x))^p dx\right)^{1/q}$$

$$\left(\int_0^{\infty} F(x)^p dx\right)^{1-1/q = 1/p} \leq \frac{p}{p-1} \|f\|_p$$

So $\|F\|_p \leq \frac{p}{p-1} \|f\|_p \quad \forall p>1, \forall f \in C_K(\mathbb{R}$

④ • First, we can prove that the Hardy's inequality holds for $f \in C_K(\mathbb{R}_+^*)$ (not necessary positive)
take $g = |f| \geq 0$

• Next, we can prove also if $(f_n)_{n\geq 0}$ be a sequence of functions in $C_K(\mathbb{R}_+)$ that Converges to f in $L^p(\mathbb{R}^+)$ then (f_n) converges locally uniformly for the anti-derivati

$$\left(F_n(x) = \int_0^x f_n(t) dt\right)_n.$$

• Finally, we use the density of $C_K(\mathbb{R}^+)$ in $L^p(\mathbb{R}_+)$
It means that:
If $f \in L^p(\mathbb{R}_+)$ there exists a sequence (f_n) in $C_K(\mathbb{R}^+)$ such that $f_n \xrightarrow[n\to\infty]{} f$ in $L^p(\mathbb{R}_+)$.

Now We take $F_n(x) = \frac{1}{x}\int_0^x f_n(t) dt$ for $x>0$, we can prove that $F_n \xrightarrow[n\to\infty]{} F(x) = \frac{1}{x}\int_0^x f(t)dt \in L^p(\mathbb{R}_+)$.

As $\forall n \geq 0$, $\|F_n\|_p \leq \frac{p}{p-1}\|f_n\|_p$, then by taking the lim
and using Fatou's lemma,
We obtain, $\|F\|_p \leq \frac{p}{p-1}\|f\|_p$. It means that:
Hardy's inequality holds for the functions in $L^p(\mathbb{R}_+)$.

⑤ If $f \equiv 0$ then Hardy inequaly holds (Trivial)

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⑥ Consider $f(x) = \chi_{[1,A]}\, x^{-1/p} = \begin{cases} x^{-1/p} & \text{on } [1,A] \\ 0 & \text{otherwise} \end{cases}$

$$\|f\|_p = \left(\int_1^A (x^{-1/p})^p\,dx\right)^{1/p} = (\ln A)^{1/p}$$

For $1 \leq x \leq A$,

$$F(x) = \frac{1}{x}\int_1^x t^{-1/p}\,dt = \frac{1}{x}\left[\frac{t^{1-1/p}}{1-1/p}\right]_1^x$$

$$= \frac{p}{p-1}\left[\left(\frac{1}{x}\right)^{1/p} - \frac{1}{x}\right]$$

For $0 < x < 1$, $F(x) = 0$ and for $x \geq A$, also $F(x) \equiv 0$

Now we suppose that exists λ such that $0 < \lambda < \frac{p}{p-1}$ and

$$\|F\|_p \leq \lambda \|f\|_p$$

We can write as $F = F_1 + F_2$ where

$$F_1(x) = \begin{cases} \frac{p}{p-1}\, x^{-1/p} & \text{on } [1,A] \\ 0 & \text{where } x < 1 \end{cases} \quad ; \quad F_2(x) = \frac{-p}{p-1}\,\frac{1}{x}\,\chi_{[1,A]}(x)$$

We know that: $\|F\|_p \geq \big|\,\|F_1\|_p - \|F_2\|_p\,\big|$

$$\|F_1\|_p - \|F_2\|_p = \frac{p}{p-1}\left(\int_1^A \frac{dx}{x} + (A^{1-1/p} - 1)^p \int_A^\infty \frac{dx}{x^p}\right)^{1/p} \cdot \left(\int_1^A \frac{dx}{x^p}\right)^{1/p}$$

$$\big|\,\|F_1\|_p - \|F_2\|_p\big| \leq \|F\|_p \leq \lambda \|f\|_p = \lambda (\ln A)^{1/p}.$$

$$\text{So} \quad \lambda \geq \frac{p}{p-1}.$$

We deduce we can not replace the constant $\frac{p}{p-1}$ by another smallest constant.