Correction list of exercices n°2

Exercise 1:

Let X be an infinite countable set.

a) Show that the set of finite subsets of X is countable.

b) Deduce that the set of infinite subsets of X is uncountable.

Solution:

We can take the case $X = \mathbb{N}$

a) The set of finite subsets of $\mathbb{N}$ is countable:

$\forall n \in \mathbb{N}$, the set X_n of subset of $\mathbb{N}$ with n elements is countable because $X_n \subset \mathbb{N}^n$.

The set of all finite subsets of $\mathbb{N}$ is $X = \bigcup_{n\in\mathbb{N}} X_n$.

It is a countable set.

b) Let Y be the set of infinite subsets of $\mathbb{N}$.

We have $\mathcal{P}(\mathbb{N}) = X \cup Y$.

If Y is a countable set then $\mathcal{P}(\mathbb{N})$ is a countable

but we know that $\mathcal{P}(\mathbb{N})$ is uncountable set.

So Y is **uncountable** set.

Exercise 2:

a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a monotonic function.

Show that the set of all discontinuity points of f is a countable set.

Hint: Consider $J(n) = \{ x \in (a,b] \mid |f(x+) - f(x-)| > 1/n \}$.

b) Same question with a monotonic function $f : \mathbb{R} \to \mathbb{R}$.

Solution:

a) We suppose that f is an increasing function.

For $y \in (a,b)$, we put $\phi(y) = \lim_{x\to y+} f(x) - \lim_{x\to y-} f(x)$.

Let $J = \{ y \in (a,b) \mid \phi(y) > 0 \}$.

J is the set of discontinuity points of f in (a,b).

We have $J = \bigcup_{n\in\mathbb{N}} J(n)$ where $J(n) = \{ y \in (a,b] \mid \phi(y) > \frac{1}{n} \}$

But $J(n)$ is finite because $\frac{1}{n} |J(n)| < f(b) - f(a)$.

So J is a countable set.

b) For $n \in \mathbb{N}$, let K_n be the set of discontinuity points of f on $(-n, n)$.

From a), K_n is a countable set. The set $K = \bigcup_{n \in \mathbb{N}} K_n$ is so countable. It is the set of all discontinuity points of f.

Exercise 3:

Remember a real number is said algebraic if it is a root of a polynomial with integers coefficients.

Show that the set of all algebraic numbers is a countable set.

Solution:

let $\mathbb{Z}[X]$ be the set of polynomial function with integers coefficients.

We have:

$$\mathbb{Z}[X] = \bigcup_{n \in \mathbb{N}} \mathbb{Z}_{\leq n}[X] \text{ where}$$

$$\mathbb{Z}_{\leq n}[X] = \{ P \in \mathbb{Z}[X] \,/\, \deg P \leq n \}.$$

$$\varphi : \mathbb{Z}_{\leq n}[X] \longrightarrow \mathbb{N}^{n+1}$$

$$P = a_0 + a_1 X + \cdots + a_n X^n \longrightarrow (a_0, a_1, \ldots, a_n) = \varphi(P)$$

φ injective (one-to-one).

So $\mathbb{Z}_{\leq n}[X]$ is a countable set. Then $\mathbb{Z}[X]$ is also a countable set.

We know that the number of roots for a polynomial function is finite.

So the set of algebraic numbers & can be written as:

$$\mathcal{A} = \bigcup_{P \in \mathbb{Z}[X]} \{ \text{roots of } P \}$$

Then $\mathcal{A}$ is a countable set.

Exercise n°4 : let X be a nonempty set. let $(A_n)_{n\geq 1}$ be a sequence of subsets of X.

a) Show that:

$$\chi_{\bigcup_{i=1}^{n} A_i} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I\subset\{1,2,\dots,n\}\\ |I|=k}} \chi_{\bigcap_{i\in I} A_i}$$

b) If X is finite, deduce Poincare's formula:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I\subset\{1,2,\dots,n\}\\ |I|=k}} \left|\bigcap_{i\in I} A_i\right|$$

Solution:

a) By induction on n:

If $n=1$, $\chi_{A_1} = \chi_{A_1}$

. $n=2$, $\chi_{A\cup B} = \chi_A(1-\chi_B)+\chi_B$.

* We suppose the result is satisfied for n: As $\bigcup_{i=1}^{n+1} A_i = \left(\bigcup_{i=1}^{n} A_i\right)\cup A_{n+1}$,

we have:

$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \chi_{\bigcup_{i=1}^{n} A_i}\left(1-\chi_{A_{n+1}}\right)+\chi_{A_{n+1}}.$$

$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \left[\sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} \chi_{\left(\bigcap_{i\in I} A_i\right)}\right]\left(1-\chi_{A_{n+1}}\right)+\chi_{A_{n+1}}.$$

We deduce that:

$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \underset{(k=1)}{\sum_{i\in\{1,\dots,n\}}} \chi_{A_i} + \sum_{k=2}^{n} (-1)^{k+1} \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} \chi_{\bigcap_{i\in I} A_i} - \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{|I|=k\\ n+1\in I}} \chi_{\bigcap_{i\in I} A_i}$$

Then, we have:

$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \sum_{i=1}^{n} \chi_{A_i} + \sum_{k=2}^{n} (-1)^{k+1} \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} \chi_{\bigcap_{i\in I} A_i} + \sum_{k=2}^{n+1} (-1)^{k+1} \sum_{\substack{|I|=k\\ n+1\in I}} \chi_{\bigcap_{i\in I} A_i}$$

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$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I \subset \{1,2,\dots,n+1\} \\ |I|=k}} \chi_{\left(\bigcap_{i\in I} A_i\right)} + (-1)^{n+2} \chi_{(A_1 \cap \dots \cap A_{n+1})}$$

$$\chi_{\bigcup_{i=1}^{n+1} A_i} = \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I|=k}} \chi_{\bigcap_{i\in I} A_i}$$

b) We take the counting measure on X (finite set) By integration we find: $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} \left|\bigcap_{i\in I} A_i\right|$

Exercise 5

Let $\mathcal{U} = \{A, B, C\}$ be a partition of X on 3 subsets. Describe the σ-algebra generated by $\mathcal{U}$.

Solution:

The σ-algebra generated by $\mathcal{U}$ is:

$$\sigma(\mathcal{U}) = \{\phi, X, A, B, C, A^c, B^c, C^c\}$$

$$= \{\phi, X, A, B, C, B \cup C, A \cup C, A \cup B\}$$

Exercise 6:

Let $\mathcal{U}$ and $\mathcal{F}$ be two σ-algebras of X. Describe the σ-algebras generated by $\mathcal{U} \cap \mathcal{F}$ and by $\mathcal{U} \cup \mathcal{F}$.

Solution:

The σ-algebra generated by $\mathcal{U} \cap \mathcal{F} = \{A \in \mathcal{P}(X),\ A \in \mathcal{U} \text{ and } A \in \mathcal{F}\}$ is $\mathcal{U} \cap \mathcal{F}$ itself.

$$\sigma(\mathcal{U} \cup \mathcal{F}) = \{A \cup B \ /\ A \in \mathcal{U} \text{ and } B \in \mathcal{F}\}$$

⚠ $\mathcal{U} \cup \mathcal{F} = \{A \in \mathcal{P}(X),\ A \in \mathcal{U} \text{ or } A \in \mathcal{F}\}$ is not closure under finite Union.

Exercise 7:

Let $\mathcal{U}$ be an algebra and μ is a measure defined on it. Let $A, B \in \mathcal{U}$. Prove that $|\mu(A) - \mu(B)| \leq \mu(A \Delta B)$

Hint: $A \subset A \cup (A \Delta B) = A \cup B$. Check whether the inequality holds if μ is an outer measure.

Solution

Since $A \subset (A \Delta B) \cup B$, by the monotonicity and semi-additivity of a measure.

$$\mu(A) \leq \mu(A \Delta B) + \mu(B)$$

Analogously

$B \subset (A \Delta B) \cup A$ then

$$\mu(B) \leq \mu(A \Delta B) + \mu(A)$$

So $\quad |\mu(A) - \mu(B)| \leq \mu(A \Delta B)$.

The properties of monotonicity and semi-additivity hold also for the outer measure μ^*, therefore

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \Delta B).$$

Exercise 8

Let X be a set, $\mathcal{U}$ an algebra of its subsets. Let $\tilde{\mathcal{U}}$ be the σ-algebra of Caratheodory measurable subsets of X. Suppose that $A \subset X$ is such that for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{U}$ such that $\mu^*(A \Delta A_\varepsilon) < \varepsilon$. Prove that $A \in \tilde{\mathcal{U}}$. (A is μ^*-measurable)

Solution:

Since any outer measure is semi-additive, it suffices to prove that for any $E \subset X$ one has

$$(I) \quad \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since $\mathcal{U} \subset \tilde{\mathcal{U}}$, one has:

$$(II) \quad \mu^*(E \cap A_\varepsilon) + \mu^*(E \cap A_\varepsilon^c) \leq \mu^*(E)$$

It follows from Ex 7,

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$$|\mu^*(E\cap A_\varepsilon)-\mu^*(E\cap A)|\leq \mu^*((E\cap A_\varepsilon)\Delta(E\cap A))$$
$$\leq \mu(A_\varepsilon \cap A) < \varepsilon$$

Then $\mu^*(E\cap A_\varepsilon) > \mu^*(E\cap A)-\varepsilon$

Analogously, $\mu^*(E\cap A_\varepsilon^c) > \mu^*(E\cap A^c)-\varepsilon$

Now we use (II), we get:

$$\mu^*(E) > \mu^*(E\cap A)+\mu^*(E\cap A^c)-2\varepsilon$$

By limit of both sides, we obtain

$$\mu^*(E) \geq \mu^*(E\cap A)+\mu^*(E\cap A^c) \quad (I)$$

Exercise 9:

For all $A\subset \mathbb{R}$ and $x\in\mathbb{R}$. Define $x.A=\{x.a \mid a\in A\}$ and

$$\mu^*(A)=\inf\left\{\sum_{n=1}^{\infty}(b_n-a_n),\ A\subset\bigcup_{n=1}^{\infty}(a_n,b_n) \,/\, (a_n,b_n) \text{ finite intervals}\right\}$$

a) Show that μ^* is an outer measure on $\mathcal{P}(\mathbb{R})$.

b) Show that $\mu^*(x.A)=|x|\,\mu^*(A)$.

c) Let $c\in\mathbb{R}$, Define $A_1=\{a\in A,\ a<c\}$ and $A_2=\{a\in A,\ a\geq c\}$. Show that

$$\mu^*(A)=\mu^*(A_1)+\mu^*(A_2).$$

Solution:

a). We must show that $\mu^*(\phi)=0$.
$\phi\subset \mathcal{U}$ for all $\mathcal{U}\in\mathcal{P}(\mathbb{R})$. Denote $\ell((a,b))=b-a$ length of interval (a,b).
So there exist $\mathcal{U}\subset\mathcal{P}(\mathbb{R})$ such that $\ell(\mathcal{U})\leq\varepsilon$, $\forall\varepsilon>0$

So $\mu^*(\phi)=\inf\{\ell(\mathcal{U}),\ \mathcal{U}\in\mathcal{P}(\mathbb{R})\}=0$

- Monotonicity of μ^*: We must show if $A \subseteq A' \subseteq \mathbb{R}$ then $\mu^*(A) \le \mu^*(A')$.

Any cover $\{U_i\}$ of A' $(A' \subset \bigcup_{i=1}^{\infty} U_i)$ is also a cover of A.
So we have:

$$\mu^*(A) = \inf\{\sum_i \ell(U_i) ; \{U_i\} \text{ a cover of } A\}$$

$$\le \inf\{\sum_i \ell(V_i) , \{V_i\} \text{ a cover of } A'\}$$

$$= \mu^*(A')$$

Where the inequality is from the definition of infimum

- Countable subadditivity of μ^*: We must show that if $\{A_i\} \subset \mathcal{P}(\mathbb{R})$ then $\mu^*(\bigcup_i A_i) \le \sum_i \mu^*(A_i)$.

Moreover, we assume that $\sum_{i=1} \mu^*(A_i) < \infty$ or we have nothing to prove. So we have:

$$\mu^*(\bigcup_i A_i) \overset{def}{=} \inf\{\sum_k \ell(E_k) ; \{E_k\} \text{ is a cover of } (\bigcup_i A_i)\}$$

$$\le \inf\{\sum_{i,j} \ell(U_j^{(i)}) ; \{U_j^{(i)}\} \text{ is a cover of } A_i\}$$

$$\| $$

$$\le \inf\{\sum_j \ell(U_j^{(1)}) ; \sum_j \ell(U_j^{(2)}), \cdots\}$$

$$\| $$

$$\le \sum_i (\inf \sum_j \ell(U_j^{(i)}))$$

$$\| $$

$$\le \sum_i \mu^*(A_i) < \infty$$

The inequality is because the set over which the infimum is taken is smaller on the right than the left hand side of the inequality.
The following equality is simply a re-ordering of the sum.