King Saud University
College of sciences
Department of mathematics
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Dr. Borhen Halouani

List of exercises $n°3$ (Math 580 Theory Measure I)

Exercise 1:

Let μ and ν be measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined as :

$$\mu = \sum_{n \geq 1} e^{-n} \delta_{1/n} \; ; \; \nu = \sum_{n \geq 1} e^{n} \delta_{1/n}.$$

1. Does the measures μ and ν are finite? probability measures? $\sigma-$finite? infinite?
2. Compute $\mu(\{0\})$; $\mu([0, \frac{1}{k}])$ for $k \geq 1$, $\lim_{k \to \infty} \mu([0, \frac{1}{k}])$ and compare the results.
3. Compute $\nu(\{0\})$, $\nu([0, \frac{1}{k}])$ for $k \geq 1$, and compare the results.

Exercise 2:

1. Let $(X, \mathcal{A})$ be a measurable space and (μ_j) be a sequence of positives measures on $\mathcal{A}$. Assume that $\forall A \in \mathcal{A}$ and $\forall j \in \mathbb{N}$, $\mu_j(A) \leq \mu_{j+1}(A)$. For all $A \in \mathcal{A}$, we put $\mu(A) = \sup_{j \in \mathbb{N}} \mu_j(A)$.

   Show that μ is a measure on $\mathcal{A}$.
2. On the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we define for all $j \in \mathbb{N}$,

$$\nu_j(A) = \mathrm{card}(A \cap [j, +\infty]).$$

   Show that ν_j is a measure on $\mathcal{P}(\mathbb{N})$ and $\nu_j(A) \geq \nu_{j+1}(A)$.
3. Let $\nu : \mathcal{P}(\mathbb{N}) \longrightarrow [0, \infty]$ defined by $\nu(A) = \inf_{j \in \mathbb{N}} \nu_j(A)$. Compute $\nu(\mathbb{N})$ and $\nu(\{k\})$ for $k \in \mathbb{N}$. Deduce that ν is not a measure.

Exercise 3:

Let λ be the Lebesgue measure on $\mathbb{R}$ and let $\varepsilon > 0$. Construct an open set Ω in $\mathbb{R}$ that is dense and $\lambda(\Omega) < \varepsilon$.

Exercise 4:

Let E be a nonempty set.

1. Let $\mathcal{E}$ be a $\sigma-$algebra on E and A be a subset of E. Show that the characteristic function χ_A is measurable if and only if $A \in \mathcal{E}$.

2. Let $\mathcal{A}$ be a partition at most countable of E, $\mathcal{E}$ be the $\sigma-$algebra generated by $\mathcal{A}$ and f be a real function on E. Show that f is measurable if and only if f is constant on each subset of $\mathcal{A}$.

3. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$ is a Borelian function.

Exercise 5:
Let E be a Borelian set of $\mathbb{R}$ and $f : E \longrightarrow \mathbb{R}$ be a monotonic function. Show that f is a measurable function.

Exercise 6:
Let $(E, \mathcal{A})$ be a measurable space and $(f_n)_n$ be a sequence of measurable functions from E to $\mathbb{R}$. Show that the following sets:
$A = \left\{x \in E,\ \lim\limits_{n\to\infty} f_n(x) = \infty\right\}$ and $B = \{x \in E, \text{the sequence } (f_n) \text{ is bounded}\}$
are measurable.

Exercise 7:
Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ be a measurable space and $f : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B})$ be a measurable function. Show that the function $\mu_f : \mathcal{B} \longrightarrow [0, \infty]$ defined by $\mu_f(B) = \mu(f^{-1}(B))$ is a measure on $\mathcal{B}$.

Exercise 8:(*Egoroff's theorem*)
Let $(X, \mathcal{A}, \mu)$ be a measure space such $\mu(X) < \infty$ and $f_n : (X, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a sequence of measurable functions.

1. Show that the set of convergence C of (f_n) is a measurable set.

2. We suppose that (f_n) is $\mu-$convergent to f a.e ($\mu(C^c) = 0$).
   For $k \in \mathbb{N}^*$, let $E_n^k = \bigcap_{k\geq n}\{|f_k - f| \leq \frac{1}{k}\}$. Show that $C \subset \bigcup_{n\geq 1} E_n^k$.
   Deduce that for all $\varepsilon > 0$, $\forall k \in \mathbb{N}^*$ there exists $n_{k,\varepsilon} \in \mathbb{N}^*$ such that
   $$\mu\left((E^k_{n_{k,\varepsilon}})^c\right) < \frac{\varepsilon}{2^k}.$$

3. Deduce that $\forall \varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{A}$ such that the sequence (f_n) converges uniformly to f on E_ε and $\mu(E_\varepsilon^c) < \varepsilon$.

4. Give a counterexample when $\mu(X) = +\infty$.

Solution of Homework n° 3

Exercise n° 1:

① $\mu(\mathbb{R}) = \sum_{n=1}^{\infty} e^{-n}\delta_{1/n}(\mathbb{R}) = \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}(e^{-1})^n$

$= \frac{1/e}{1-1/e} = \frac{1}{e-1} < 1$

μ is finite and σ-finite but not of probability.

$\nu(\mathbb{R}) = \sum_{n=1}^{\infty} e^{n}\delta_{1/n}(\mathbb{R}) = \sum_{n=1}^{\infty} e^{n} = \infty \quad , \quad e>1$

ν is infinite.

Take $A_k = (-\infty, 0] \cup [1/k, \infty)$; $\bigcup_k A_k = \mathbb{R}$

$\nu(A_k) = \sum_{n=1}^{\infty} e^{n}\delta_{1/n}([1/k, \infty)) = \sum_{n=1}^{k} e^{n} < \infty, \forall k \geq 1$

because $\frac{1}{n} \in A_k$ iff $n \leq k$. So ν is σ-finite and not of probability.

② $\mu(\{0\}) = \sum_{n=1}^{\infty} e^{-n}\delta_{1/n}(\{0\}) = 0$

For $k \geq 1$, $\mu([0, 1/k]) = \sum_{n=1}^{\infty} e^{-n}\delta_{1/n}([0, 1/k]) = \sum_{n=k}^{\infty} e^{-n} = \frac{e^{-k}}{1-e^{-1}}$

Then

$$\mu(\{0\}) = \lim_{k\to\infty} \mu([0, 1/k]) = 0$$

$\left(A_k = [0, 1/k] \downarrow, \ \mu\text{-finite} \ \mu([0,1]) < \infty \right).$

③ $\nu(\{0\}) = \sum_{n=1}^{\infty} e^{n}\delta_{1/n}(\{0\}) = 0$

$\nu([0, 1/k]) = \sum_{n=1}^{\infty} e^{n}\delta_{1/n}([0, 1/k]) = \sum_{n=k}^{\infty} e^{n} = +\infty$

for $k \in \mathbb{N}^*$,

So $0 = \nu(\{0\}) \neq \lim_{k\to\infty} \nu([0, 1/k]) = +\infty$

ν is not finite.

Exercise n° 2.

① - We have $\mu(\phi) = 0$.
- Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint sequence of elements of $\mathcal{A}$.

we put $A = \bigcup_{n\geq 1} A_n$

For all j and n, we have: $\mu_j(A) \geq \mu_j(A_1) + \cdots + \mu_j(A_n)$
We take the limit of both sides when $j\to\infty$

$$\mu(A) \geq \mu(A_1) + \cdots + \mu(A_n)$$

By limit when $n\to\infty$,

$$\mu(A) \geq \sum_{n\geq 1} \mu(A_n) \quad (1)$$

If $\sum_{n\geq 1} \mu(A_n) = \infty$ then $\mu(A) = \infty$.

We suppose that $\sum_{n\geq 1} \mu(A_n) < \infty$. Then $\forall \varepsilon > 0$, $\exists i$ such that

$$\sum_{n>i} \mu(A_i) \leq \varepsilon \text{. So } \forall j, \quad \mu_j(A) \leq \mu_j(A_1) + \cdots + \mu_j(A_i) + \varepsilon \quad (*)$$

because μ_j is a measure.
When $j\to\infty$, $(*)$ becomes $\mu(A) \leq \mu(A_1) + \cdots + \mu(A_i) + \varepsilon$

So $\mu(A) \leq \sum_{n\geq 1} \mu(A_n) + \varepsilon$

As ε is arbitrary,

we get $\mu(A) \leq \sum_{n\geq 1} \mu(A_n) \quad (2)$

From (1) and (2), we deduce

$$\mu(A) = \sum_{n\geq 1} \mu(A_n)$$

②
$$\nu_j : \mathcal{P}(\mathbb{N}) \longrightarrow [0,\infty]$$
$$A \longrightarrow \nu_j(A) = \operatorname{card}(A \cap [j,\infty])$$

. $\nu_j(\phi) = \operatorname{card}\{\phi\} = 0$
. Let $(A_n)_n$ be a disjoint sequence of subsets of $\mathbb{N}$

$$\nu_j\left(\bigcup_{n\geq 1} A_n\right) = \operatorname{Card}\left(\bigcup_{n\geq 1} A_n \cap [j,\infty]\right)$$
$$= \operatorname{Card} \bigcup_{n\geq 1} (A_n \cap [j,\infty])$$
$$= \sum_{n\in\mathbb{N}^*} \operatorname{card}(A_n \cap [j,\infty]) \underset{\text{are disjoint}}{\overset{\text{because}}{=}} \sum_{n\geq 1} \nu_j(A_n).$$

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③ $\forall j$, $\text{card}(\mathbb{N}\cap[j,\infty]) = \infty$, So $\nu(\mathbb{N}) = \infty$.
If $k\in\mathbb{N}$, there exists $j\in\mathbb{N}$ such that $j>k$ and $\{k\}\cap[j,\infty] = \phi$. We deduce $\nu(\{k\}) = 0$.
If ν is a measure then $\underbrace{\nu(\mathbb{N})}_{\infty} = \sum_{k\in\mathbb{N}} \underbrace{\nu(\{k\})}_{0}$ ↯.

Exercise n°3:
We know that $\mathbb{Q}$ is dense in $\mathbb{R}$.
We enumerate $\mathbb{Q}$, $\{q_1, q_2, \dots\}$
We take $\Omega = \bigcup_{n\geq 1}]q_n - \varepsilon/2^{n+2}, q_n + \varepsilon/2^{n+2}[$. We have $\lambda(\Omega) < \varepsilon$

Exercise n°4:
① for all Borelian subset of $\mathbb{R}$, $\chi_A^{-1}(B)$ is A or A^c or E if B that contains 1 and not 0, 0 and not 1 or $\{0,1\}$ respectively.
So χ_A is measurable if and only if $A\in\mathcal{E}$

②
We suppose that f is constant on each $A\in\mathcal{A}$.
Put a_A the value of f on each $A\in\mathcal{A}$. Then
$f = \sum_{A\in\mathcal{A}} a_A \chi_A$. From ①, χ_A is a measurable function then f is measurable. As $\mathcal{A}$ is at most countable, f is the limit of measurable sequence of functions.
So f is measurable.
Conversely, we suppose that f is measurable but there exists a subset $A\in\mathcal{A}$ and 2 elements of A which f takes 2 distincts values. Denote it y and z.
Consider now $B = A\cap\{f=y\}$ and $C = A\cap\{f=z\}$.
B and C are not empty and are disjoints in $\mathcal{E}$.
In particular B and C are the union of subsets of $\mathcal{A}$.
But B and C have an intersection with A that contains some elements. ↯ (Contradiction because $\mathcal{A}$ is a partition of E)
③ 1st method : Take $g : \mathbb{R} \longrightarrow [-\infty,\infty]$

(g_n) is a sequence of continuous functions.

$\mathbb{R}^* = (-\infty, 0) \cup (0, \infty)$ is an open set then it is a Borelian set of $\mathbb{R}$.

(h_n) converges simply to f on $\mathbb{R}$ then f is a Borelian function.

Exercise 5 :

We can assume that f is increasing function.

We can take the interval $[a, +\infty)$.

let $b = \inf f^{-1}([a, +\infty))$.

Then $f^{-1}([a, \infty)) \overset{?}{=} (b, +\infty) \cap X$ or $[b, \infty) \cap X$.

1) $f^{-1}([a, \infty)) \subset [b, +\infty) \cap X$?

by construction, b is the inf of all elements of $f^{-1}([a, \infty))$. Hence $f^{-1}([a, +\infty)) \subset [b, +\infty) \cap X$.

2) Conversely let $x \in (b, \infty) \cap X$.

by definition of b, there exists $y \in f^{-1}([a, +\infty))$ / $b < y < x$. As f is increasing then $f(y) \le f(x)$.

But $y \in f^{-1}([a, +\infty))$ then $f(y) \ge a$. We deduce that $f(x) \ge a$. We obtain $f^{-1}([a, \infty)) \supset (b, \infty) \cap X$

Exercise 6 :

$$A = \{x \in E, \lim_{n \to \infty} f_n(x) = \infty\} = \bigcap_{\substack{A > 0 \\ A \in \mathbb{Q}}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \ge N} \{x \in E, f_n(x) > A\}$$

As $\{x \in E, f_n(x) > A\}$ are Borelian then A is Borelian

$$B = \{x \in E, (f_n) \text{ bounded}\} = \bigcup_{\substack{a \in \mathbb{Q} \\ b \in \mathbb{Q}}} \bigcap_{n \in \mathbb{N}} \{x \in E, a < f_n(x) < b\}$$

is also a Borelian set.

Exercise 7 :

. $\mu(f^{-1}(\phi)) = \mu(\phi) = 0$ because μ is a measure.

. let $(A_i)_{i \in I}$ be a disjoint sequence of elements of $\mathcal{B}$.

(I = countable set)

$$\mu\left(f^{-1}\left(\bigcup_{i \in I} A_i\right)\right) = \mu\left(\bigcup_{i \in I} f^{-1}(A_i)\right)$$

As $(f^{-1}(A_i))_{i \in I}$ is a disjoint sequence of elements of

As μ is a measure on $\mathcal{A}$ then $\mu\left(\bigcup_{i\in I} f^{-1}(A_i)\right) = \sum_{i\in I} \mu(f^{-1}(A_i))$

So $\mu_f\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} \mu_f(A_i)$

We deduce that μ_f is a measure on $\mathcal{B}$.

Exercise 8: (Egoroff's theorem).

① $C = \{x\in X,\ (f_n(x))_n \text{ is convergent}\}$

$$= \bigcap_{\substack{\varepsilon>0\\ \varepsilon\in\mathbb{Q}}} \ \bigcup_{\substack{a,b\in\mathbb{R}\\ a<b<a+\varepsilon\\ N\in\mathbb{N}}} \ \bigcap_{n\geq N} \{x\in X,\ a<f_n(x)<b\}$$

As $\{x\in X,\ a<f_n(x)<b\}$ are measurable sets and we know that the union of measurable sets and intersection countable of measurable sets is a measurable. ($C\in\mathcal{A}$)

② let $x\in C$ then $\forall\varepsilon>0\ \exists N$ such that $\forall n\geq N$, $|f_n(x)-f(x)|<\varepsilon$

We take $\varepsilon = 1/k$ and then $x\in E_N^k$. So $C\subset \bigcup_{n\geq1} E_n^k$.

As $f_i - f$ is measurable, thes sets E_n^k are measurable.

We have $C\subseteq \bigcup_{n\geq1} E_n^k$ and so

$\bigcap_{n\geq1} (E_n^k)^c \subseteq C^c$, we have so $\lim_{n\to\infty} \mu((E_n^k)^c) = 0$

because $\mu(C^c)=0$ and μ is finite.

Then for $\varepsilon/2^k$, $\exists n_{k,\varepsilon}$ such that $\forall n\geq n_{k,\varepsilon}$,

$\mu((E_n^k)^c) < \varepsilon/2^k$. It follows $\mu((E_{n_{k,\varepsilon}}^k)^c) < \varepsilon/2^k$

③ We take $E_\varepsilon = \bigcap_{k\geq1} E_{n_{k,\varepsilon}}^k \in\mathcal{A}$ and

$$\mu(E_\varepsilon^c) = \mu\left(\bigcup_{k\geq1} (E_{n_{k,\varepsilon}}^k)^c\right) \leq \sum_{k\geq1} \mu((E_{n_{k,\varepsilon}}^k)^c) < \sum_{k\geq1} \frac{\varepsilon}{2^k} \leq \varepsilon.$$

So $\mu(E_\varepsilon^c) \leq \varepsilon$.

④ let $X=[0,\infty)$, $\mathcal{A}$ be the σ-algebra trace on X of $\mathcal{B}(\mathbb{R})$, μ be the Lebesgue measure and $f_n = \chi_{A_n}$ where $A_n=[n,\infty)$. Then

$f_n \xrightarrow[n\to\infty]{} f\equiv 0$ but $f_n \not\xrightarrow[\text{uniformly}]{} 0$.