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Solution of list of Exercises n°5 (Math 580)
Theory Measure

Exercise n°1: f is monotonic on $[a,b]$, the problem is symmetric.

① We can suppose that f is increasing ($\forall x \geq y \Rightarrow f(x) \geq f(y)$)

let $C = f(b) - f(a)$ and $n > 2$.

For $k \in [\![0, n-2]\!]$, we set

$$I_k = \left\{ x \in [a,b],\ f(a) + \frac{kc}{n} < f(x) < f(a) + \frac{(k+1)c}{n} \right\}$$

and $I_{n-1} = \left\{ x \in [a,b],\ f(a) + \frac{(n-1)c}{n} < f(x) < f(a) + c = f(b) \right\}$

If $x < y \in I_k$ and $z \in [x,y]$ then $f(x) \leq f(z) \leq f(y)$

So $z \in I_k$. Then I_k is an open interval.

All these intervals (I_k) are disjoints pairwise and $\bigcup_{k=0}^{n-1} I_k = [a,b]$.

- let g_n (resp. h_n) the step functions defined as:

$$g_n(x) := f(a) + \frac{k.c}{n} \quad \text{on } I_k$$

$$\left(\text{resp. } h_n(x) := f(a) + \frac{(k+1)c}{n}\right)$$

We have

$$g_n \leq f \leq h_n \text{ on } I_k \text{ for all } k$$

and $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = f$ pointwise on $[a,b]$

We deduce that f is Riemann-integrable.

② let $x \in [a,b)$. Consider the sequence $\left(x_n = x + \frac{1}{n}\right)_n$.

$(x_n) \downarrow$ (decreasing) and converges to x.

f is increasing, $(f(x_n))_n$ is decreasing and

$\mathbb{R} \ni f(a) \leq f(x_n)\ \forall n$. Then $f(x_n)$ has a limit ℓ when n tends to ∞.

Now we want to show that $\lim_{y\to x^+} f(y) = \ell$

let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow |f(x_n) - \ell| \leq \varepsilon$

We set $\delta = x_N - x = \frac{1}{N} > 0$.

If $|y - x| < \delta$ and $y > x$, we have $x < y < x_N$

As f is increasing, $f(y) < f(x_N)$. It follows that

$|f(y) - \ell| \leq |f(x_N) - \ell| \leq \varepsilon$ So $\lim f(y) = \ell$.

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Exercise n°2

① We use the definition of Riemann integrability:
$\forall \varepsilon > 0$, $\exists$ 2 step functions f_ε and m_ε on $[a,b]$ to $\mathbb{R}$
such that
$$\begin{cases} |f - f_\varepsilon| \leq m_\varepsilon \\ \int_a^b m_\varepsilon \, dx \leq \varepsilon . \end{cases}$$
Let $\varepsilon > 0$,
we have $\big||f| - |f_\varepsilon|\big| \leq |f - f_\varepsilon| \leq m_\varepsilon$

for a fixed $\varepsilon > 0$, the existence of m_ε and $|f_\varepsilon|$ show that $|f|$ is Riemann-integrable. And if f is a step function we have $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)| dx$ by computing the area of rectangle under the graph of f.
– Let (f_n) be a sequence of step functions that converges to f for $\varepsilon = 1/n$,
$$\left|\int_a^b f(x)\,dx\right| = \lim_{n\to\infty} \left|\int_a^b f_n(x)\,dx\right|$$
$$\leq \lim_{n\to\infty} \int_a^b |f_n(x)|\,dx = \int_a^b |f|\,dx$$

② We assume that $f \geq 0$.
Let $\varepsilon > 0$. There exist 2 step function φ_ε and ψ_ε such that
$$|f - \varphi_\varepsilon| \leq \psi_\varepsilon \quad \text{and} \quad \int \psi_\varepsilon < \varepsilon .$$
We can assume that $\varphi_\varepsilon \geq 0$ (because $\big||f| - |\varphi_\varepsilon|\big| \leq |f - \varphi_\varepsilon|$)
We remark that
φ_ε^p and ψ_ε^p are also step functions.
By Mean Value thm; $\forall (u,v) \in [a,b]^2$ $\exists\, z \in (u,v)$ such that
$$u^p - v^p = p\, z^{p-1}(u - v)$$
Let $M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} \varphi_\varepsilon\}$. Then $\forall x \in [a,b]$,
$$|f^p(x) - \varphi_\varepsilon^p(x)| \leq p M^{p-1} |f(x) - \varphi_\varepsilon(x)|$$
$$\leq p M^{p-1} \psi_\varepsilon(x).$$
Now we Remplace ε by $\frac{\varepsilon}{pM^{p-1}}$ in the choice of ψ_ε $\left(\text{i.e. } \int_a^b \psi_\varepsilon < \frac{\varepsilon}{pM^{p-1}}\right)$.

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③ We have: $f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

The sum of 2 Riemann-integrable functions is a Riemann-integrable

By ② with $p=2$, the square of a Riemann-integrable function is also a Riemann-integrable. So we deduce that the product of 2 Riemann integrable functions is a Riemann-Integrable function

Exercise n°3:

Let $m \geq n$. We have

$$\int_0^1 |f_m - f_n|\,dx = \int_0^1 \left(\sum_{k=n+1}^{m} \frac{x^k}{k}\right) dx = \sum_{k=n+1}^{m} \frac{1}{k(k+1)}$$

$$\Rightarrow \int_0^1 |f_m - f_n|\,dx = \sum_{k=n+1}^{m}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{n+1} - \frac{1}{m+1} \leq \frac{1}{n+1}$$

$$\lim_{n,m\to\infty} \int_0^1 |f_m - f_n|\,dx = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So $(f_n)_{n\geq 1}$ is a Cauchy sequence for the norm N_1.

Now we assume that f_n converges to $f \in \mathcal{C}([0,1],\mathbb{R})$ for the norm N_1.

Let $M = \sup_{[0,1]} f(x)$. We know that the series $\sum_k \frac{x^k}{k}$ diverges when $x = 1$.

Let $0 < \eta < 1$ such that for $n > N$, the partial sums of $f_n \geqslant M+1$ on $]\eta, 1]$. It follows that:

On $]\eta,1]$, we have: $|f_n - f| > 1$, $\forall n > N$.

then $\int_\eta^1 |f_n - f|\,dx \underset{n\to\infty}{\not\longrightarrow} 0$ $\lightning$

so $f \notin \mathcal{C}([0,1],\mathbb{R})$

Exercise 4:

Let x, y and z 3 points in $\mathbb{R}^2$ defined as:

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$x = (0,0)$; $y = (1,0)$ and $z = (0,1)$.

Consider $\mu = \delta_x + \delta_y + \delta_z$ (Sum of Dirac on $\mathbb{R}^2$ at x, y and z).

We suppose that there exist 2 measures α and β on $\mathbb{R}$ such that $\mu = \alpha \otimes \beta$

Writting now the measure of elements, pairs of $\{x, y, z\}$, we see that $\alpha(\{0\}) = \alpha(\{1\}) = \beta(\{0\}) = \beta(\{1\}) = 1$

But $\mu(\{1,1\}) = 0 \neq \alpha(\{1\}) \cdot \beta(\{1\}) = 1$

So μ can not be the tensorial product of 2 measures on $(\mathbb{R}, \mathcal{B})$.

Exercise n° 5:

① The σ-algebra $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) \subset \mathcal{P}(\mathbb{N}^2)$

Conversely, $\mathbb{N}^2$ is a countable set. All subset A of $\mathbb{N}^2$ is the Union of $\{(m,n)\}_{\substack{m \in \mathbb{N} \\ n \in \mathbb{N}}}$ at most countable.

But $\{(m,n)\} = \{m\} \times \{n\}$.

So $\{(m,n)\} \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$

It follows that: $\mathcal{P}(\mathbb{N}^2) \subset \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$.

We deduce that: $\mathcal{P}(\mathbb{N}^2) = \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$

② let $A \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$, we have:

$$\mu \otimes \mu (A) = \sum_{(m,n) \in A} \mu \otimes \mu(\{(m,n)\}).$$

$$\mu \otimes \mu (A) = \sum_{(m,n) \in A} 1 = \mathrm{Card}\,(A).$$

$\mu \otimes \mu$ is the counting measure of $\mathbb{N}^2$

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Exercise n° 6 ① let $(x,y) \in [0,1]^2$, the projections

$\pi_y : (x,y) \longrightarrow x$ and $\pi_x : (x,y) \longrightarrow y$ are measurable, from $([0,1]^2, \mathcal{B} \otimes \mathcal{P})$ to $([0,1], \mathcal{B})$ and $([0,1], \mathcal{P})$ respectively.

So $(x,y) \xmapsto{f} (x-y)$ is a measurable function (continuous)

$\Delta = f^{-1}(\{0\})$; then $\Delta = \{(x,x) \,/\, x \in [0,1]\}$ is a measurable set

So $\Delta \in \mathcal{B} \otimes \mathcal{P}$.

② For $y \in [0,1]$, $\int \chi_\Delta(x,y)\, d\lambda(x) = \lambda(\{y\}) = 0$

(λ: Lebesgue measure)

So $\int d\mu(y) \left(\int \chi_\Delta(x,y)\, d\lambda(x) \right) = 0$

In other hand, For $x \in [0,1]$, $\int \chi_\Delta(x,y)\, d\mu(y) = \mu(\{x\}) = 1$

(μ: counting measure)

then $\int d\lambda(x) \left(\int \chi_\Delta(x,y)\, d\mu(y) \right) = 1$.

③ χ_Δ is a measurable function positive, we can not applied Tonelli-Fubini's thm because μ is not σ-finite.

Exercise 7

① $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{x^2+y^2}\, dy \right) dx =$

$\int_0^1 \frac{x^2-y^2}{x^2+y^2}\, dy = \int_0^1 \left[\frac{x^2+y^2-2y^2}{x^2+y^2} \right] dy = \int_0^1 \left[1 - 2\frac{y^2}{x^2+y^2} \right] dy$

$= \int_0^1 dy - 2\int_0 \frac{y^2}{x^2+y^2}\, dy$

$\int_0^1 \frac{y^2}{x^2+y^2}\, dy = \frac{1}{x^2} \int_0^1 \frac{y^2}{1+(y/x)^2}\, dy = \frac{1}{x^2} \int_0^{1/x} \frac{(xu)^2\, x\, du}{1+u^2} = x \int_0^{1/x} \frac{u^2}{1+u^2}\, du$

$u = y/x \Rightarrow du = \frac{dy}{x}$

$= x \left[\int_0^{1/x} \frac{1+u^2-1}{1+u^2}\, du \right] = x \left[\int_0^{1/x} \left(1 - \frac{1}{1+u^2} \right) du \right]$

$= x \left[1/x - \tan^{-1}\left(\frac{1}{x}\right) \right] = 1 - x \tan^{-1}\left(\frac{1}{x}\right)$

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$$\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy = 1-2\left[1-x\tan^{-1}\left(\frac{1}{x}\right)\right] = 2x\tan^{-1}\left(\frac{1}{x}\right)-1$$

So $$\int_0^1\left(\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy\right)dx = \int_0^1\left[2x\tan^{-1}\left(\frac{1}{x}\right)-1\right]dx$$
$$= 2\left(\int_0^1 x\tan^{-1}\left(\frac{1}{x}\right)dx\right)-1.$$

$$\int_0^1 x\tan^{-1}\left(\frac{1}{x}\right)dx = \left[\frac{x^2}{2}\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

Integration by parts: $u'(x)=x$, $u(x)=x^2/2$

$v(x)=\tan^{-1}\left(\frac{1}{x}\right) \Rightarrow v'(x)=\dfrac{-1/x^2}{1+(1/x)^2}$

$$\int_0^1 x\tan^{-1}\left(\frac{1}{x}\right)dx = \frac{1}{2}\tan^{-1}(1)+\frac{1}{2}\int_0^1 \frac{(x^2+1)-1}{1+x^2}\,dx$$
$$= \frac{1}{2}\tan^{-1}(1)+\frac{1}{2}\left[\int_0^1 dx - \int_0^1 \frac{dx}{1+x^2}\right]$$

$$\int_0^1 x\tan^{-1}\left(\frac{1}{x}\right)dx = \frac{1}{2}\tan^{-1}(1)+\frac{1}{2}\left[1-\left[\tan^{-1}(x)\right]_0^1\right] = 1/2$$

$$\int_0^1\left(\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy\right)dx = 0$$

- $$\int_0^1\left(\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx\right)dy = ?$$

$$\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx = \int_0^1 \frac{x^2+y^2-2y^2}{x^2+y^2}\,dx = \int_0^1\left(1-\frac{2y^2}{x^2+y^2}\right)dx$$
$$= 1-2\int_0^1 \frac{y^2}{y^2+x^2}\,dx = 1-2\int_0^1 \frac{dx}{1+\left(\frac{x}{y}\right)^2}$$
$$= 1-2y\int_0^{1/y}\frac{du}{1+u^2} = 1-2y\tan^{-1}\left(\frac{1}{y}\right)$$ $u=x/y$, $du=\frac{dx}{y}$

$$\int_0^1\left(\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx\right)dy = \int_0^1\left[1-2y\tan^{-1}\left(\frac{1}{y}\right)\right]dy$$
$$= 1-2\int_0^1 y\tan^{-1}\left(\frac{1}{y}\right)dy = 1-2\times\frac{1}{2}=0$$

$$\int_0^1\left(\int_0^1 f_1(x,y)\,dy\right)dx = \int_0^1\left(\int_0^1 f_1(x,y)\,dx\right)dy = 0.$$

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$$\cdot \int_0^1 \left(\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}} \, dy \right) dx =$$

$$\int_0^1 \frac{(x-y)\, dy}{(x^2+y^2)^{3/2}} = x \int_0^1 \frac{dy}{(x^2+y^2)^{3/2}} - \frac{1}{2} \int_0^1 2y \,(x^2+y^2)^{-3/2} dy$$

$$\int_0^1 2y \,(x^2+y^2)^{-3/2} dy = \left[[x^2+y^2]^{-1/2} \right]_0^1 = \frac{1}{\sqrt{x^2+1}} - \frac{1}{x}, \quad x>0$$

$$\int_0^1 \frac{dy}{(x^2+y^2)^{3/2}} = \int_0^1 \frac{dy}{\left[x^2(1+(\frac{y}{x})^2)\right]^{3/2}} = \frac{1}{x^3} \int_0^1 \frac{dy}{\left[1+(y/x)^2\right]^{3/2}}$$

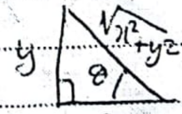
$$\frac{y}{x} = \tan\theta$$

$$\frac{1}{x} dy = \sec^2\theta \, d\theta$$

$$\int_0^1 \frac{dy}{\left[1+(y/x)^2\right]^{3/2}} = x \int_0^{\tan^{-1}\frac{1}{x}} \frac{\sec^2\theta \, d\theta}{\sec^3\theta} = x \int_0^{\tan^{-1}(1/x)} \cos\theta \, d\theta$$

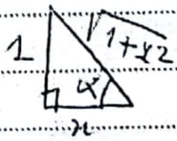
$$= x \left[\sin\theta \right]_0^{\tan^{-1}(\frac{1}{x})}$$

$$= x \sin\left(\tan^{-1}(1/x)\right)$$



$$\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}} \, dy = \frac{1}{x} \sin\Big(\underbrace{\tan^{-1}(1/x)}_{\alpha} \Big) - \frac{1}{2} \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right)$$

$$= \frac{1}{x} \cdot \frac{1}{\sqrt{1+x^2}} - \frac{1}{2} \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right)$$



$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}} \, dy \right) dx = \int_0^1 \left(\frac{1}{x\sqrt{1+x^2}} - \frac{1}{2\sqrt{1+x^2}} + \frac{1}{2x} \right) dx$$

$$= \left[-\operatorname{csch}^{-1}(x) - \frac{1}{2} \sinh^{-1}(x) + \frac{1}{2} \ln x \right]_0^1 = +\infty$$

$$= -\ln(1+\sqrt{2}) - \tfrac{1}{2} \ln(1+\sqrt{2}) + \lim_{x\to 0} \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right) - \tfrac{1}{2} \lim_{x\to 0} \ln x = +\infty$$

$\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$ $\qquad \sinh^{-1}(0) = 0$; $\sinh^{-1}(1) = \ln(1+\sqrt{2})$

$\operatorname{csch}^{-1}(x) = \ln\left(\frac{1+\sqrt{x^2+1}}{|x|}\right)$; $\lim_{x\to 0} \operatorname{csch}^{-1}(x) = -\infty$; $\operatorname{csch}^{-1}(1) = \ln(1+\sqrt{2})$

$$\cdot \int_0^1 \left(\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}} \, dx \right) dy$$

$$\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}} \, dx = \frac{1}{2} \int_0^1 2x\,(x^2+y^2)^{-3/2} dx - y \int_0^1 \frac{dx}{(x^2+y^2)^{3/2}}$$

$$\cdot \frac{1}{2} \int_0^1 2x\,(x^2+y^2)^{-3/2} dx = \frac{1}{2} \left[[x^2+y^2]^{-1/2} \right]_0^1 = \frac{1}{2} \left(\frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right), \; y>0$$

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$$\int_0^1 \frac{dx}{(x^2+y^2)^{3/2}} = \frac{1}{y^3}\int_0^1 \frac{dx}{\left(1+\left(\frac{x}{y}\right)^2\right)^{3/2}}$$

$$\int_0^1 \frac{dx}{\left(1+\left(\frac{x}{y}\right)^2\right)^{3/2}} = y\,\frac{1}{\sqrt{1+y^2}}$$

So $\displaystyle\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}}\,dx = \frac{1}{2}\left(\frac{1}{\sqrt{1+y^2}}-\frac{1}{y}\right) - y\cdot\frac{1}{y^3}\cdot y\,\frac{1}{\sqrt{1+y^2}}$

$$= \frac{1}{2\sqrt{1+y^2}} - \frac{1}{2y} - \frac{1}{y\sqrt{1+y^2}}$$

$$\int_0^1\left(\int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}}\,dx\right)dy = \int_0^1\left[\frac{1}{2\sqrt{1+y^2}} - \frac{1}{2y} - \frac{1}{y\sqrt{1+y^2}}\right]dy$$

$$=\left[\frac{1}{2}\sinh^{-1}(y) - \frac{1}{2}\ln y + \operatorname{csch}^{-1}(y)\right]_0^1$$

$$=\left[\frac{1}{2}\ln(1+\sqrt{2}) + \ln(1+\sqrt{2}) + \frac{1}{2}\lim_{y\to 0}\ln y - \lim_{y\to 0}\ln\left(\frac{1+\sqrt{y^2+1}}{y}\right)\right]$$

$$= -\frac{1}{2}\lim_{y\to 0}\ln y = +\infty$$

$$\int_0^1\int_0^1 f_2(x,y)\,dx\,dy = \int_0^1\int_0^1 f_2(x,y)\,dy\,dx = +\infty$$

② f_1 and f_2 don't satisfy Fubini-Tonelli's Thm because they are not positives (but measurable and the lebesgue measure is σ-finite on $[0,1]$). But the two integrals are equal (finite or not).

Exercise n°8:

① Let $\phi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$

$(x, y) \longmapsto f(x) - y$

As f is a Borelian function then ϕ is also Borelian function

Moreover $\Gamma = \phi^{-1}(\{0\})$. So Γ is a Borelian set of $\mathbb{R}^{d+1}$.

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② χ_Γ is a Borelian, positive function and let λ be the Lebesgue measure on $\mathbb{R}^{d+1}$ (its is σ-finite measure)

By Fubini's Thm, we have:

$$\lambda(\Gamma) = \int_{\mathbb{R}^{d+1}} \chi_\Gamma(x_1, \dots, x_d, y)\, dx_1 \dots dx_d\, dy$$

$$\lambda(\Gamma) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \chi_\Gamma(x_1, \dots, x_d, y)\, dy \right) dx_1 \dots dx_d$$

$$\lambda(\Gamma) = \int_{\mathbb{R}^d} \underbrace{dy(\{f(x_1, \dots, x_d)\})}_{=0} dx_1 \dots dx_d$$

$$\lambda(\Gamma) = \int_{\mathbb{R}^d} 0 \, dx_1 \dots dx_d = 0$$

So Γ is a null set for the Lebesgue measure of $\mathbb{R}^{d+1}$.

⚠ This generalize the fact: . the length of a point is zero ($d=0$)
. the surface of a curve is zero ($d=1$)
. the volume of a surface is zero ($d=2$)

Exercise n°9

① The function $f(x,y) = \dfrac{1}{(1+x+y)^\alpha}$ is a Borelian, positive function. By Fubini's Thm, $\displaystyle\int_{[0,\infty]^2} f(x,y)\,dx\,dy = \int_0^\infty \left(\int_0^\infty \frac{dx}{(1+x+y)^\alpha} \right) dy$

* For all $y \geq 0$; $\displaystyle\int_0^\infty \frac{dx}{(1+x+y)^\alpha} < \infty$ if and only if $\boxed{\alpha > 1}$

So f is not integrable if $\alpha \leq 1$.

* If $\alpha > 1$, for every $y \geq 1$, we have $\displaystyle\int_0^\infty \frac{dx}{(1+x+y)^\alpha} = \frac{1}{\alpha - 1}(1+y)^{1-\alpha}$

But the function $y \longrightarrow (1+y)^{1-\alpha}$ is integrable if and only if $\alpha - 1 > 1 \Leftrightarrow \boxed{\alpha > 2}$

So By Fubini's Thm, f is integrable if and only if $\alpha > 2$ and

$$\int_{[0,\infty)^2} \frac{1}{(1+x+y)^\alpha}\, dx\, dy = \frac{1}{(\alpha-1)(\alpha-2)}$$

② $\frac{1}{x} = \int_0^\infty e^{-xt}\, dt$. It follows by linearity of integral, that

$$\int_0^a \frac{\sin x}{x}\, dx = \int_0^a \left(\int_0^\infty (\sin x)\, e^{-xt}\, dt \right) dx$$

– The function $(x,t) \longrightarrow e^{-xt} \sin x$ is integrable on $[0,a] \times [0,\infty)$

because
$$\int_{[0,a]\times[0,\infty)} |e^{-xt} \sin x|\, dt\, dx = \int_0^a \left(\int_0^\infty |e^{-xt} \sin x|\, dt \right) dx$$
$$= \int_0^a \frac{|\sin x|}{x}\, dx.$$

and $x \longrightarrow \frac{|\sin x|}{x}$ is continuous on $[0,a]$. So it's bounded. Hence it is integrable.

By Fubini's thm,

$$\int_0^a \frac{\sin x}{x}\, dx = \int_0^\infty \left(\int_0^a e^{-xt} \sin x\, dx \right) dt$$

But
$$\int_0^a e^{-xt} \sin x\, dx = \frac{1 - e^{-at}(\cos a + t \sin a)}{1+t^2}$$

(Double Integration by parts)

We set $f_a(t) = \frac{1 - e^{-at}(\cos a + t\sin a)}{1+t^2}$, $\lim_{a\to\infty} f_a(t) = ?$

For $a \geq 1$, $|f_a(t)| \leq \frac{1}{1+t^2}\left(1 + e^{-t}(1+t)\right) \leq \frac{3}{1+t^2}$ $\forall t > 0$

$$f_a(t) \xrightarrow[a\to\infty]{} \frac{1}{1+t^2}$$ pointwise are on $[0,\infty)$

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As $\int_0^\infty \frac{dt}{1+t^2} < \infty$. Then By Dominate Convergence Thm

$$\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx = \int_0^\infty \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^\infty = \frac{\pi}{2}$$

$$\text{So } \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2} \quad (\text{Dirichlet integral})$$

$\int_0^\infty \frac{\sin x}{x}\, dx$ is not absolute convergent because $\int_0^\infty \frac{|\sin x|}{x} dx = \infty$

* $\frac{\sin x}{x}$ is not integrable. By Contradiction,

As $|\sin x| \geq \sin^2 x$ then $\frac{\sin^2 x}{x} = \frac{1-\cos 2x}{2x}$.

if $\frac{\sin x}{x}$ is integrable then $\frac{\sin^2 x}{x}$ is integrable but

$\frac{1}{2x}$ is not integrable on $(0,\infty)$ then we deduce

that $\frac{\sin x}{x}$ is not integrable

(4) $f(x,y) = x^y$ is continuous on $[0,1]\times[a,b]$. So

f is a Borelian function and positive.

By Tonelli-Fubini's Thm.

$$\int_{[0,1]\times[a,b]} x^y\, dx\, dy = \int_a^b \left(\int_0^1 x^y dx\right) dy = \int_a^b \frac{1}{y+1} dy$$

$$= \ln\left(\frac{b+1}{a+1}\right). \text{ So } f \text{ is integrable}$$

Also $\int_{[0,1]\times[a,b]} x^y\, dx\, dy = \int_0^1 \left(\int_a^b x^y\, dy\right) dx = \ln\left(\frac{b+1}{a+1}\right)$.

As $\int_a^b x^y\, dy = \frac{x^b - x^a}{\ln x}$ is dx-lebesgue measure on $(0,1]$ and a.e $[0,1]$

$x \to \frac{x^b - x^a}{\ln x}$ is integrable on $[0,1]$.

By Fubini's Thm, we get: $\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln\left(\frac{b+1}{a+1}\right)$.

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