Solutions to Math 204 Mid I(36/37)S2 (Exam held on: 21-05-1437; March 1, 2016)

Solution to Question 1

(a) $\frac{dy}{dx} = \frac{xe^x}{(1-y^2)} = f(x,y)$. Then $\frac{\partial f}{\partial x}(x,y) = \frac{-xe^x(-2y)}{(1-y^2)^2} = \frac{2xye^x}{(1-y^2)^2}$. $\frac{dy}{dx} = f(x,y)$, $y(0) = 0$ has unique solution on the region containing $(0,0)$ whence f and $\frac{\partial f}{\partial y}$ are continuous. f and $\frac{\partial f}{\partial y}$ are continuous on $\{(x,y)\colon y < -1\} \bigcup \{(x,y)\colon -1 < y < 1\} \bigcup \{(x,y)\colon y > 1\}$.
It follows that the requested region is: $\{(x,y)\colon -1 < y < 1\}$.
(b) Here $P(x) = -2x$ and $Q(x) = e^x(1-2x)$. Integrating factor: $\psi(x) = e^{\int P(x)dx} = e^{\int -2xdx} = e^{-x^2}$. So, $\int \psi(x)Q(x)dx = \int e^{-x^2}e^x(1-2x)dx = \int(1-2x)e^{x-x^2}dx = e^{x-x^2} + C$. Hence the final solution is $y(x) = e^{x^2}\left(e^{x-x^2} + C\right)$, i.e., $y(x) = e^x + ce^{x^2}$.

Solution to Question 2

(a) Here $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = (1+\frac{2}{y})\cos x \Leftrightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the equation is not exact. We have $\frac{(M_y - N_x)}{N} = -\cot x$. So the integrating factor: $e^{-\int \cot x dx} = \csc x$. Let $M = \cos x \csc x = \cot x$ and $N = (1+\frac{2}{y})\sin x \csc x = 1 + \frac{2}{y}$, so that $M_y = 0 = N_x$. Now from $\frac{\partial y}{\partial x} = \cot x$, we get $f(x,y) = \ln(\sin x) + h(y)$ whence $h'(y) = 1 + \frac{2}{y}$ and $h(y) = y + \ln y^2$. Hence the solution of the given differential equation is $\ln(\sin x) + y + \ln y^2 = C$.
(b) Here $f(x,y) = \frac{x}{y} + \frac{y}{x}$. Then $f(tx,ty) = \frac{tx}{ty} + \frac{ty}{tx} = f(x,y)$ implies that f is homogeneous. Now let $u = \frac{y}{x}$, we have $\frac{dy}{dx} = x\frac{du}{dx} + u$, $\frac{x}{y} + \frac{y}{x} = u + \frac{1}{u}$ implying that $\frac{du}{dx} = (\frac{1}{u})(\frac{1}{x})$,i.e., $udu = \frac{dx}{x}$ gives $\frac{u^2}{2} = \ln|x| + C \Rightarrow u^2 = 2\ln|x| + C$. So, $\frac{y^2}{x^2} = 2\ln|x| + C$. Since $y(1) = 2$, $C = 4$, the solution is $y^2 = x^2(2\ln|x| + 4)$.

Solution to Question 3

Diving both sides of the given differential equation by y^3, one obtains: $y^{-3}\frac{dy}{dx} + (\frac{1}{2}\tan x)y^{-2} = \frac{(4x+5)^2}{2\cos x}$. Letting $u = y^{-2}$, we have $\frac{du}{dx} - (\tan x)u = -\frac{(4x+5)^2}{\cos x}$. Integrating factor is: $e^{\int -\tan x dx} = e^{\int -\frac{\sin x}{\cos x}dx} = e^{\ln|\cos x|} = \cos x$. Thus, we have $u = \frac{1}{\cos x}\int -\cos x\frac{(4x+5)^2}{\cos x}dx + \frac{C}{\cos x} \Rightarrow u\cos x = -\frac{1}{12}(4x+5)^3 + C$, i.e., $\frac{1}{y^2} = -\frac{1}{12\cos x}(4x+5)^3 + \frac{C}{\cos x}$.

Solution to Question 4

Here $T_m = 10^oF$. So, we have the DE: $\frac{dT}{dt} = k(T-10) \Rightarrow T(t) = 10 + ce^{kt}$. As $T(0) = 70$, one obtains: $70 = 10 + ce^0$ implies $c = 60^oF$ so that $T(t) = 10 + 60e^{kt}$. Also, as given $T(\frac{1}{2}) = 50^oF$, we get $50 = 10 + 60e^{\frac{k}{2}} \Rightarrow k = 2\ln(\frac{4}{6})$. Hence $T(t) = 10 + 60e^{2\ln(\frac{4}{6})t}$. Now at $t = 1$, we get $T(1) = 10 + 60e^{\ln(\frac{16}{36})} = 10 + 60(\frac{16}{36}) = 10 + 26.6 = 36.7^oF$. If, then $T(t) = 15^o$, we get $15 = 10 + 60e^{\ln(\frac{16}{36})t} \Rightarrow t = \frac{\ln(\frac{1}{12})}{\ln(\frac{16}{36})} = 3.06$ min.