King Saud University
College of sciences
Department of mathematics
Second semester 1431/1432 H

Dr. Borhen Halouani
Time: 3 hours
Full Marks: 50

Final Exam Math 580 (Theory Measure I)

Exercise 1:(15 points)

Let f_n, f be positives, measurables functions on a measure space $(X, \mathcal{A}, \mu)$ and $f \in L^1$. We suppose that:

$$f \leq \liminf_{n \to +\infty} f_n \quad \mu - a.e \quad \text{and } \limsup_{n \to +\infty} \int f_n d\mu \leq \int f d\mu.$$

1. Prove that $\int f d\mu \leq \liminf_{n \to +\infty} \int f_n d\mu$. Deduce that
$$\lim_{n \to +\infty} \int f_n d\mu = \int f d\mu.$$
2. Show that $f = \liminf_{n \to +\infty} f_n \quad \mu - a.e.$
3. Prove that $\forall a, b \in \mathbb{R}$, $|a - b| = a + b - 2\min\{a, b\}$ and deduce that (f_n) converges to f in L^1.
4. Give an example of sequence (f_n) to show that (f_n) does not necessary convergent to $f \quad \mu - a.e.$

Exercise 2:(15 points)

Let $f : [0, 1] \longrightarrow [0, +\infty)$ be a Borel function and $A \subset \mathbb{R}^3$ be the set

$$A := \{(x, y, z)/0 \leq x \leq 1 \; ; \; y^2 + z^2 \leq f(x)\}.$$

1. Prove that the function $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F(x, y, z) = y^2 + z^2 - f(x)$ is a Borel function and deduce that $A \in \mathcal{B}(\mathbb{R}^3)$.
2. For every $x \in [0, 1]$, determine the section $A_x := \{(y, z)/(x, y, z) \in A\}$. Verify that $A_x \in \mathcal{B}(\mathbb{R}^2)$ and compute its Lebesgue measure $\lambda_2(A_x)$.
3. Compute the volume $\lambda_3(A)$ of A as function of f.

4. Verify that $\lambda_3(A) = \frac{\pi}{4}$ if $f(x) = x^3$, $\forall x \in [0,1]$.

Problem:(20 points)

Let $\mathbb{R}_+ = [0, +\infty)$ and for $n \geq 1$, the function $f_n : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{C}$ defined by

$$f_n(t,x,y) := e^{-(n+x)y}\left(e^{iyt} - 1\right).$$

1. Show that $\forall (t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times (0, +\infty)$,

$$\sum_{n\geq 1} |f_n(t,x,y)| \leq |t| \frac{ye^{-y}}{1 - e^{-y}}$$

and $f(t,x,y) := \sum_{n=1}^{+\infty} f_n(t,x,y) = \frac{e^{-xy}}{e^y - 1}\left(e^{iyt} - 1\right)$.

2. Show that for every $(t,x) \in \mathbb{R} \times \mathbb{R}_+$, the function $y \longmapsto f(t,x,y) \in L^1(\mathbb{R}_+, \lambda)$ and

$$\int_0^{+\infty} f(t,x,y)dy = \sum_{n\geq 1} \frac{it}{(n+x-it)(n+x)}.$$

3. For $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, let $u(t,x,y) := \frac{e^{-xy}}{e^y - 1}\sin(yt)$. Show that $\forall (t,x) \in \mathbb{R} \times \mathbb{R}_+$, the function $y \longmapsto u(t,x,y) \in L^1(\mathbb{R}_+, \lambda)$ and

$$\int_0^{+\infty} \frac{e^{-xy}}{e^y - 1}\sin(yt)dy = \sum_{n\geq 1} \frac{t}{(n+x)^2 + t^2}.$$

4. Show that the function $g : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by

$$g(t,x) := \int_0^{+\infty} \frac{e^{-xy}}{e^y - 1}\sin(yt)dy$$

is of class $C^2(\mathbb{R} \times \mathbb{R}_+)$[1] and satisfying the equation $\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g}{\partial x^2} = 0$ on $\mathbb{R} \times \mathbb{R}_+$.

[1]The class $C^2(\mathbb{R} \times \mathbb{R}_+)$ is the set of all functions f twice differentiable and f'' is continuous on $\mathbb{R} \times \mathbb{R}_+$.

Solution Final exam Math 580 (Theory Measure)

Second Semester 1431/1432H

Exercise n° 1 (15 points)

① Since $f_n \geq 0$, we apply Fatou's lemma:

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

By hypothesis, we have $f \leq \liminf_n f_n$. So

4 $$\int f \, d\mu \leq \int \liminf f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu \quad (*)$$

As $\liminf_{n\to\infty} \int f_n \, d\mu \leq \limsup_{n\to\infty} \int f_n \, d\mu$, we obtain

$$\int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu \leq \limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$$

These inequalities become equality and we get in particular,

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu \quad (**)$$

②

As $f \leq \liminf_{n\to\infty} f_n$ μ-a.e and from

3 $(*)$ and $(**)$ we get $\int f \, d\mu = \int \left(\liminf_{n\to\infty} f_n\right) d\mu$

We deduce that $f = \liminf_{n\to\infty} f_n$ μ-a.e

③ If $a \geq b$, then $a + b - 2\min\{a,b\} = a + b - 2b$

$= a - b = |a-b|$

If $a \leq b$, then $a + b - 2\min\{a,b\} = a + b - 2a$

2 $= b - a = |a-b|$

So $\forall a,b \in \mathbb{R}$, $|a-b| = a + b - 2\min\{a,b\}$.

It follows that:

$$\int |f_n - f|\, d\mu = \int f_n\, d\mu + \int f\, d\mu - 2\int \min\{f_n, f\}\, d\mu$$

But

$$\liminf_{n\to\infty} \min\{f_n, f\} = \min\{\liminf_{n\to\infty} f_n , f\} = f \ \mu\text{-a.e}$$

Applying Fatou's lemma, we get:

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$$\int f\, d\mu \leq \liminf_{n\to\infty} \int \min\{f_n, f\}\, d\mu$$

and

$$\limsup_{n\to\infty} \int |f_n - f|\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu + \int f\, d\mu - \liminf_{n\to\infty} 2\int \min\{f_n, f\}\, d\mu$$

$$\leq 2\int f\, d\mu - 2\int f\, d\mu = 0$$

So $\lim\limits_{n\to\infty} \int |f_n - f|\, d\mu = 0.$

4) It suffices to find a sequence (f_n) of measurable functions, positives such that $f_n \xrightarrow[n\to\infty]{} 0$ in L^1 μ-a.e $\left(\lim\limits_{n\to\infty} \int |f_n - 0|\, d\mu = 0\right)$ But

$$\liminf_{n\to\infty} f_n = 0 \ \mu\text{-a.e} \quad \text{and} \quad \limsup_{n\to\infty} f_n > 0.$$

We take $f_{2^n+k} := \chi_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]}$ for all $k \in \{0, \dots, 2^n - 1\}$ $\forall\, n \in \mathbb{N}$

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$$\liminf_{n\to\infty} f_n = \sup_n \{\inf_{k\geq n} f_k\} = 0$$

$$\limsup_{n\to\infty} f_n = \inf_n \{\sup_{k\geq n} f_k\} = 1$$

Take $([0,1], \mathcal{B}([0,1]), \mu)$ as Lebesgue measure space.

$$\int \chi_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}\, d\mu = \int_{k/2^n}^{k+1/2^n} d\mu(x) = \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$$

$f_n \longrightarrow 0$ in L^1

Exercise n° 2: (15 points)

① $F_1(x,y,z) = y^2+z^2$ (Polynomial function) is continuous and $F_2(x,y,z) = f(x)$ is a Borel function. So

6 $F = F_1 + F_2$ is also a Borel function.

We have
$$A = F^{-1}\left((-\infty, 0]\right) = \{(x,y,z) / y^2+z^2-f(x) \leq 0\}$$
So

A is a Borel Set.

② If $x \in [0,1]$ then $A_x = \{(y,z) / y^2+z^2 \leq f(x)\}$

3 is the Closed disc of center $(0,0)$ and radius $r = \sqrt{f(x)} > 0$

A_x is bounded, closed in $\mathbb{R}^2$ and $\lambda_2(A_x) = \pi f(x)$.

③ By Fubini's Theorem, we have $\lambda_3 = \lambda_1 \otimes \lambda_1 \otimes \lambda_1 = \lambda_1 \otimes \lambda_2$

4 So $\lambda_3(A) = \int_0^1 \lambda_2(A_x)\,dx = \int_0^1 \pi f(x)\,dx$
$$= \pi \int_0^1 f(x)\,dx.$$

④ If $f(x) = x^3$;

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$$\lambda_3(A) = \pi \int_0^1 x^3\,dx = \frac{\pi}{4}.$$

Problem (20 points)

① For all $n \geq 1$, we have: For all $y \geq 0$.

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$$|f_n(t,x,y)| = \left| e^{-(n+x)y}\left(e^{iyt}-1\right)\right| \leq e^{-ny}|t|y.$$

because:
$$\left|e^{iyt}-1\right|^2 = \left(e^{iyt}-1\right)\left(e^{-iyt}-1\right) = 1 - \big(\underbrace{e^{iyt}+e^{-iyt}}_{2\cos yt}\big)+1$$
$$= 2(1-\cos yt) \leq (yt)^2$$

$\forall u \in \mathbb{R}$, $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots$; So $\cos u \geq 1 - \frac{u^2}{2}$ $\forall u \in \mathbb{R}$; $(u = yt)$

We get for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$

$$\sum_{n\geq 1} |f_n(t,x,y)| \leq |t| y \sum_{n\geq 1} e^{-ny} = |t| y \sum_{n\geq 1} (e^{-y})^n$$ (geometric series)

$$\leq |t| \, y \, \frac{e^{-y}}{1-e^{-y}}.$$

③ This series is absolutely convergent (likes ..., ...)

for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times (0,\infty)$ and

$$f(t,x,y) = \sum_{n\geq 1} f_n(t,x,y) = \frac{e^{-xy}}{e^y - 1}(e^{iyt} - 1).$$

$$\left(\sum_{n\geq 1}^{\infty} e^{-ny} = \frac{e^{-y}}{1-e^{-y}} = \frac{1}{e^y - 1} \right)$$

geometric series $e^{-y} < 1, \forall y > 0$.

② For $(t,x) \in \mathbb{R} \times \mathbb{R}_+$,

$$\int_0^{+\infty} |f(t,x,y)| \, dy \leq \int_0^{\infty} \sum_{n\geq 1} |f_n(t,x,y)| \, dy$$

$$\leq |t| \int_0^{\infty} \frac{y e^{-y}}{1-e^{-y}} \, dy$$

② $$\leq |t| \int_0^{\infty} \frac{y}{e^y - 1} \, dy < \infty$$

$$\left(\text{because } \lim_{y\to\infty} \frac{y}{e^y} = 0 \text{ and } \lim_{y\to 0} \frac{y}{e^y - 1} = \lim_{y\to 0} \frac{1}{e^y} = 1 \right)$$

Also, we have:

$$\int_0^{\infty} |f_n(t,x,y)| \, dy \leq |t| \cdot \int_0^{\infty} y e^{-ny} \, dy = \frac{|t|}{n^2}.$$

(by integration by parts)

$$\int_0^{+\infty} y e^{-ny} \, dy = \frac{1}{n^2} \qquad u(y) = y \;;\; u'(y) = 1, \quad v'(y) = e^{-ny} \;;\; v(y) = -\frac{1}{n} e^{-ny}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series with $p = 2 > 1$ is convergent)

We deduce that: $\sum_{n \geq 1} \| f_n(t, x, \cdot) \|_{L^1(\mathbb{R}_+, \lambda)} < \infty$

We can also apply Montone Convergence.

$$\sum_{n=1}^{\infty} \left(\int_0^{+\infty} |f_n(t,x,y)| \, dy \right) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} |f_n(t,x,y)| \right) dy$$

$$\leq \int_0^{\infty} |t| \frac{y e^{-y}}{1 - e^{-y}} \, dy < +\infty$$

By The theorem of absolutely convergent series, we find

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$$\int_0^{+\infty} f(t,x,y) \, dy = \sum_{n=1}^{\infty} \left(\int_0^{\infty} f_n(t,x,y) \, dy \right)$$

$$\int_0^{+\infty} f_n(t,x,y) \, dy = \int_0^{+\infty} e^{-(n+x)y} e^{iyt} \, dy - \int_0^{+\infty} e^{-(n+x)y} \, dy$$

$$\int_0^{+\infty} f_n(t,x,y) \, dy = \int_0^{+\infty} e^{-[(n+x) - it]y} \, dy - \int_0^{+\infty} e^{-(n+x)y} \, dy$$

$$\int_0^{\infty} f_n(t,x,y) \, dy = \frac{1}{(n+x) - it} - \frac{1}{(n+x)} = \frac{it}{(n+x-it)(n+x)}$$

because $y \longrightarrow e^{-(n+x)y} e^{iyt}$ and $y \longrightarrow e^{-(n+x)y}$ are integrables on $\mathbb{R}_+$.

So we get:

$$\int_0^{+\infty} f(t,x,y) \, dy = \sum_{n=1}^{\infty} \left(\frac{it}{(n+x-it)(n+x)} \right).$$

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③ Note that for all $(t,x,y) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$

$$u(t,x,y) = \frac{e^{-xy}}{e^y-1}\sin yt = \operatorname{Im} f(t,x,y)$$ ②

$$\left(f(t,x,y) = \frac{e^{-xy}}{e^y-1}\left(e^{iyt}-1\right) = \frac{e^{-xy}}{e^y-1}\left[\cos(yt)+i\sin(yt)-1\right] \right)$$

So $\|u(t,x,\cdot)\|_{L^1(\mathbb{R}_+,\lambda)} \leq \|f(t,x,\cdot)\|_{L^1(\mathbb{R}_+,\lambda)}$ ①

We deduce that $y \mapsto u(t,x,y) \in L^1(\mathbb{R}_+,\lambda)$

Moreover, $\displaystyle\int_0^{+\infty} u(t,x,y)\,dy = \sum_{n=1}^{\infty} \operatorname{Im}\left(\frac{it}{(n+x-it)(n+x)}\right)$

$$\frac{it}{(n+x-it)(n+x)} = \frac{it(n+x+it)}{\left((n+x)^2+t^2\right)(n+x)} = \frac{-t^2+it(n+x)}{\left((n+x)^2+t^2\right)(n+x)}$$ ②

Then $\operatorname{Im}\left(\dfrac{it}{(n+x-it)(n+x)}\right) = \dfrac{t}{(n+x)^2+t^2}$.

We deduce that:

$$\int_0^{+\infty} u(t,x,y)\,dy = \sum_{n=1}^{\infty}\frac{t}{(n+x)^2+t^2}.$$

④ It is clear that $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+\times(0,\infty))$.
Here we can apply the theorem of differentiability for integral depending on parameter

It suffices to find a function $h \in L^1(\mathbb{R}_+, \lambda)$ such that

$$\left(\left|\frac{\partial u}{\partial t}\right| + \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial^2 u}{\partial t^2}\right| + \left|\frac{\partial^2 u}{\partial t \partial x}\right| + \left|\frac{\partial^2 u}{\partial x^2}\right| \right)(t,x,y) \leq h(y)$$

for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times (0,\infty)$.

By computing, we have:

$*$ $\dfrac{\partial u}{\partial t}(t,x,y) = \dfrac{e^{-xy}}{e^y - 1}\, y \cos yt$; $*$ $\dfrac{\partial u}{\partial x} = -y \dfrac{e^{-xy}}{e^y-1} \sin yt$

$*$ $\dfrac{\partial^2 u}{\partial t^2}(t,x,y) = -\dfrac{e^{-xy}}{e^y-1}\, y^2 \sin yt$; $*$ $\dfrac{\partial^2 u}{\partial t \partial x}(t,x,y) = -\dfrac{e^{-xy}}{e^y-1}\, y^2 \cos yt$

$*$ $\dfrac{\partial^2 u}{\partial x^2}(t,x,y) = y^2 \dfrac{e^{-xy}}{e^y-1} \sin(yt)$ 3

We can take $h(y) = 3(y+y^2)\dfrac{e^{-y}}{1-e^{-y}}$, $y>0$

(because $x \geq 0$)

$$\|h\|_{L^1(\mathbb{R}_+,\lambda)} = \int_0^{+\infty} 3(y+y^2)\frac{e^{-y}}{1-e^{-y}}\,dy < +\infty$$

We deduce that

$$g(t,x) = \int_0^{+\infty} u(t,x,y)\,dy \in C^2(\mathbb{R}\times\mathbb{R}_+)$$ 2

and for all $(t,x) \in \mathbb{R}\times\mathbb{R}_+$,

$$\left(\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g}{\partial x^2}\right)(t,x) = \int_0^{\infty} \left(\underbrace{\frac{\partial^2 u}{\partial t^2}(t,x,y) + \frac{\partial^2 u}{\partial x^2}(t,x,y)}_{=0} \right) dy$$

$\Delta g(t,x) = 0$. It means that g is a harmonic function on $\mathbb{R}\times\mathbb{R}_+$.