Question: 1 . (a) Given

$$\begin{aligned} x + y + z &= 0 \\ y + 3z &= 5 \\ -x - 2y + 2z &= 7 \end{aligned}$$ **[10]**

i. **Use method of cofactors to find A^{-1}, where A is coefficient matrix, and**
ii. **Use A^{-1} to solve the given system**

Soln.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ -1 & -2 & 2 \end{vmatrix} = 6$$

Matrix of cofactors $C = \begin{bmatrix} 8 & -3 & 1 \\ -4 & 3 & 1 \\ 2 & -3 & 1 \end{bmatrix}$

$$\operatorname{Adj} A = C^T = \begin{bmatrix} 8 & -4 & 2 \\ -3 & 3 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{6}\begin{bmatrix} 8 & -4 & 2 \\ -3 & 3 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 8 & -4 & 2 \\ -3 & 3 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -20+14 \\ 15-21 \\ 5+7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -6 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$x = -1, \quad y = -1, \quad z = 2.$

(b) Use properties of the determinants (without expanding) to show that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(c-a)(b-a)(c-b)$$ **[6]**

Soln.

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= abc\,(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= abc\,(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} \; R_3 - R_2$$

$$= abc\,(b-a)(c-a)(c-b)$$

(c) Find the value of $\det\left((3A)^2\left(2A^{-1}\right)^{-1}\right)$ where A is $3X3$ matrix and $\det(A)=2$. **[4]**

Soln

$\det\ (3A \cdot 3A \cdot (2A^{-1})^{-1})$

$= \det 3A \cdot \det 3A \cdot \det(2A^{-1})^{-1}$

$= 3^3 \cdot \det A \quad 3^3 \det A \cdot \frac{1}{2^3} \det A$

$= (27)(2) \cdot (27)(2) \cdot \frac{1}{2^3}(2)$

$= (27)(27) = 729$

Question: 2 .(a) Find equation of the plane containing points A(1,2,3), B(2,3,1) and C(3,1,2). Also find distance of the plane from the point D(2,3,5). **[6]**

Sdn. $\vec{AB} = \langle 1, 1, -2 \rangle$, $\vec{AC} = \langle 2, -1, -1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} = -3i - 3j - 3k$$

Equation of plane containing A, B, and C is

$-3(x-1) - 3(x-2) - 3(x-3) = 0$

$-3x - 3y - 3z + 18 = 0$

Distance of point D(2, 3, 5) from the plane is

$$d = \frac{|-3(2) - 3(3) - 3(5) + 18|}{\sqrt{9+9+9}} = \frac{12}{\sqrt{27}}$$

(b) Let $r(t)=(2t^3-3)i+(5t^2+3)j+(8t+2)k$ **be a vector valued function, find the parametric equation of the tangent line at point P** $(-1,8,10)$. **[6]**

Soln

$r'(t) = \langle 6t^2, 10t, 8 \rangle$ Finding value of t

$8t + 2 = 10$

$8t = 8$

$t = 1$

$r'(1) = \langle 6, 10, 8 \rangle$

Equation of line through point P(-1, 8, 10) with parallel vector $r'(1) = \langle 6, 10, 8 \rangle$ is

$x = -1 + 6t$

$y = 8 + 10t$

$z = 10 + 8t, \quad t \in R.$

Question: 3 . (a) A particle starts at an initial position $r(0) = <1,1,1>$ with initial velocity $v(0) = 0$. Its acceleration is $a(t) = i + 2tj + 3t^2k$. Find its velocity, and position at time t. [10]

Soln. $a(t) = i + 2tj + 3t^2 k$ and $r(0) = <1,1,1>$

$v(0) = <0,0,0>$

$$v(t) = \int a(t)\,dt = ti + t^2 j + t^3 k + c_1$$

Since $v(0) = 0 \Rightarrow c_1 = 0$

$$v(t) = ti + t^2 j + t^3 k$$

$$r(t) = \int v(t)\,dt = \frac{t^2}{2} i + \frac{t^3}{3} j + \frac{t^4}{4} k + c$$

Since $r(0) = <1,1,1> \Rightarrow c = <1,1,1>$

$$r(t) = \left(\frac{t^2}{2} + 1\right) i + \left(\frac{t^3}{3} + 1\right) j + \left(\frac{t^4}{4} + 1\right) k.$$

(b) Let $r(t) = <t^2, t, \frac{t^2}{2}>$ be the position vector of a moving point P at time t.

Find tangential and normal components of acceleration at anytime t. Also find curvature κ. [10]

Soln

$r'(t) = <2t, 1, t>$

$r''(t) = <2, 0, 1>$

$|r'| = \sqrt{5t^2+1}$, $r' \cdot r'' = 5t$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 2t & 1 & t \\ 2 & 0 & 1 \end{vmatrix} = i - 2k$$

$|r' \times r''| = \sqrt{1+4} = \sqrt{5}$

Tangential component of acceleration

$$a_T = \frac{r' \cdot r''}{|r'|} = \frac{5t}{\sqrt{5t^2+1}}$$

Normal component of acceleration

$$a_N = \frac{|r' \times r''|}{|r'|} = \frac{\sqrt{5}}{\sqrt{5t^2+1}}$$

Curvature

$$\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{\sqrt{5}}{[5t^2+1]^{3/2}}$$

Question: 4 . (a) Show that the function $z(x,t)=\sin(n\pi x)\cos(na\pi t)$

satisfies the wave equation $\frac{\partial^2 z}{\partial t^2}-a^2\frac{\partial^2 z}{\partial x^2}=0$. **[8]**

Soln

$$\frac{\partial z}{\partial t} = -na\pi \sin(n\pi x)\sin(na\pi t)$$

$$\frac{\partial^2 z}{\partial t^2} = -n^2a^2\pi^2 \sin(n\pi x)\cos(na\pi t) \longrightarrow 1$$

$$\frac{\partial z}{\partial x} = n\pi\cos(n\pi x).\cos(na\pi t)$$

$$\frac{\partial^2 z}{\partial x^2} = -n^2\pi^2\sin(n\pi x).\cos(na\pi t)$$

$$-a^2\frac{\partial^2 z}{\partial x^2} = a^2n^2\pi^2\sin(n\pi x).\cos(na\pi t) \longrightarrow 2$$

Adding Eq. 1 and Eq. 2

$$\frac{\partial^2 z}{\partial t^2} - a^2\frac{\partial^2 z}{\partial x^2} = 0.$$

(b) Let $w=4x^2+4y^2+z^2$, where $x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta, z=\rho\cos\phi$

Show that $\frac{\partial w}{\partial \theta}=0.$ **[8]**

Soln.

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= 8x.(-\rho\sin\phi\sin\theta) + 8y.(\rho\sin\phi\cos\theta) + 2z.(0)$$

$$= -8(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta) + 8(\rho\sin\phi\sin\theta)(\rho\sin\phi\cos\theta)$$

$$= -8\rho^2\sin^2\phi\cos\theta\sin\theta + 8\rho^2\sin^2\phi\sin\theta\cos\theta = 0$$

Question: 5 (a) Find the directional derivative of $f(x,y,z)=\sqrt{(x+2y+3z)^3}$

at the point P(1, 1, 2) in the direction of the vector $a=2j-k$. **[8]**

Soln

$$\nabla f = \left\langle \tfrac{3}{2}(x+2y+3z)^{\frac{1}{2}},\ \tfrac{3}{2}(x+2y+3z)^{\frac{1}{2}}.2,\ \tfrac{3}{2}(x+2y+3z)^{\frac{1}{2}}.3 \right\rangle$$

$$\nabla f(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle \qquad |a| = \sqrt{4+1} = \sqrt{5}$$

$$u = \left\langle 0, \tfrac{2}{\sqrt{5}}, \tfrac{-1}{\sqrt{5}} \right\rangle$$

$$D_u f(1,1,2) = 0 + \frac{18}{\sqrt{5}} - \frac{27}{2\sqrt{5}} = \frac{9}{2\sqrt{5}}$$

(b) Find the point of paraboloid $z = x^2 + y^2$ at which normal line is perpendicular to line passing through points P(1 , 2 , 1) and Q(2 , 3 , 2), if x = 2. [8]

Soln. $F(x,y,z) = x^2 + y^2 - z = 0$

$n = \nabla f = \langle 2x, 2y, -1 \rangle$

$\vec{PQ} = \langle 1, 1, 1 \rangle$

Normal line is perpendicular to the line if $n \cdot \vec{PQ} = 0$

$n \cdot \vec{PQ} = 2x + 2y - 1 = 0$

To find point if $x = 2$.

$4 + 2y - 1 = 0, \quad 2y + 3 = 0, \quad y = \frac{3}{2}$

$z = x^2 + y^2 = 4 + \frac{9}{4} = \frac{25}{4}$

Required point is $(2, \frac{3}{2}, \frac{25}{4})$.

Question: 6.(a) Find the relative extrema and saddle points, if any, of the function

$$f(x, y) = x^3 - 3x + 3xy^2.$$

[8]

Soln. $f_x = 3x^2 - 3 + 3y^2, \quad f_y = 6xy, \quad f_{xx} = 6x, \quad f_{yy} = 6x, \quad f_{xy} = 6y$

critical points: $f_x = 0 \Rightarrow 3x^2 - 3 + 3y^2 = 0 \Rightarrow x^2 + y^2 = 1$

$f_y = 0 \Rightarrow 6xy = 0 \Rightarrow x = 0$, or $y = 0$

For $x = 0$, $y^2 = 1, \Rightarrow y = \pm 1$

For $y = 0$, $x^2 = 1 \Rightarrow x = \pm 1$

Required point are $(0, 1), (0, -1), (+1, 0), (-1, 0)$

$D = f_{xx} f_{yy} - f_{xy}^2 = 36x^2 - 36y^2 = 36(x^2 - y^2)$

$D(0,1) = -36 < 0$ saddle point

$D(1,0) = 36 > 0$ Local extrema

$D(0,-1) = -36 < 0$ saddle point

$D(-1,0) = 36 > 0$ Local extrema

At point $(1, 0)$, $f_{xx} = 6 > 0$ Local minimum

At point $(-1, 0)$, $f_{xx} = -6 < 0$ Local maximum.

Local minima

$f(1,0) = -2$

Local maxima

$f(-1,0) = 2.$

(b) Use Lagrange multipliers to find the maximum value of the function $f(x,y) = 4xy$ subject to constraint $x^2 + y^2 - 1 = 0$. **[8]**

Soln

$\nabla f(x,y) = \lambda g(x,y)$

$\nabla f(x,y) = \langle 4y, 4x \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\langle 4y, 4x \rangle = \lambda \langle 2x, 2y \rangle$

$= \langle 2\lambda x, 2\lambda y \rangle$

$\Rightarrow$ $4y = 2\lambda x$ $\quad$ $2y = \lambda x$ $\quad$ $\lambda = \frac{2y}{x}$

$4x = 2\lambda y$ $\quad \Rightarrow \quad$ $2x = \lambda y$ $\quad$ $\lambda = \frac{2x}{y}$

$x^2 + y^2 = 1$ $\quad$ $x^2 + y^2 = 1$

$\Rightarrow \frac{2y}{x} = \frac{2x}{y} \Rightarrow 2y^2 = 2x^2 \Rightarrow y^2 = x^2$

$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$

$y^2 = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}}$

Values of x and y are set of points

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 2$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2$

$f(x,y)$ takes maximum value at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f(x,y)$ takes minimum value at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$