**King Saud University**
**College of Sciences**
**Mathematics Department**

**Academic Year (G) 2017–2018**
**Academic Year (H) 1438–1439**
**Bachelor AFM: M. Eddahbi**

**Solution of the final exam ACTU 362 May 13, 2018**

**Problem 1. (8 marks)** You are given the following information:

$$\ell_1 = 9700, \quad q_1 = q_2 = 0.020, \quad q_4 = 0.026 \text{ and } d_3, = 232$$

1. **(2 marks)** Determine the expected number of survivors to age 5.

Given the following portion of a life table:

| $x$ | $\ell_x$ | $d_x$ | $p_x$ | $q_x$ |
|---|---|---|---|---|
| 0 | 1000 | | 0.875 | |
| 1 | | | | |
| 2 | 750 | | | 0.25 |
| 3 | | | | |
| 4 | | | | |
| 5 | 200 | 120 | | |
| 6 | | | | |
| 7 | | 20 | | 1 |

2. **(2 marks+1 bonus)** Determine the value of the product $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6$

You are given the following life table function

| $x$ | 40 | 41 | 42 | 43 | 44 | 45 |
|---|---|---|---|---|---|---|
| $\ell_x$ | 10000 | 9900 | 9700 | 9400 | 9000 | 8500 |

3. **(2 marks)** Calculate (a) ${}_{2.6}q_{41}$ (b) ${}_{1.6}q_{40.9}$ assuming uniform distribution of deaths between integral ages.

4. **(2 marks)** Calculate (a) ${}_{2.6}q_{41}$ (b) ${}_{1.6}q_{40.9}$ assuming constant force of mortality between integral ages.



**Solution of problem 1.**

1. We recursively compute $\ell_x$ through $x = 5$. We

$$q_x = 1 - \frac{\ell_{x+1}}{\ell_x} \iff \ell_{x+1} = \ell_x(1 - q_x)$$

then we have

$$\begin{aligned} \ell_2 &= \ell_1(1 - q_1) = 9700(1 - 0.020) = 9506 \\ \ell_3 &= \ell_2(1 - q_2) = 9506(1 - 0.020) = 9315.88 \\ \ell_4 &= \ell_3 - d_3 = 9315.88 - 232 = 9083.88 \\ \ell_5 &= \ell_4(1 - q_4) = 9083.88(1 - 0.026) = 8847.70 \end{aligned}$$

2. We are asked to compute

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6 = {}_5p_1 q_{1+5} = {}_{5|}q_1$$

so we have

$${}_{5|}q_1 = \frac{\ell_6 - \ell_7}{\ell_1}$$

We knonw $\ell_7 = 20$, since everyone dies that year. We calculate

$$\ell_6 = \ell_5 - d_5 = 80 \text{ and } \ell_1 = 1000(0.875) = 875.$$

Therefore

$${}_{5|}q_1 = \frac{80 - 20}{875} = \frac{12}{175} = 0.0685$$

3. Under **UDD**: We have by definition

   (a)

$$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_{2}p_{41} \times_{0.6} p_{43} = 1 - {}_{2}p_{41}\left(1 -_{0.6} q_{43}\right) \\ &= 1 - \frac{\ell_{43}}{\ell_{41}}\left(1 - 0.6\left(1 - \frac{\ell_{44}}{\ell_{43}}\right)\right) = 1 - \frac{9400}{9900}\left(1 - 0.6\left(1 - \frac{9000}{9400}\right)\right) = 0.074747 \end{aligned}$$

   (b) We have ${}_{1.6}q_{40.9} = 1 - {}_{1.6}p_{40.9}$ and

$$\begin{aligned} {}_{1.6}p_{40.9} &= \frac{{}_{2.5}p_{40}}{{}_{0.9}p_{40}} = \frac{{}_{2}p_{40} \times {}_{0.5}p_{42}}{{}_{0.9}p_{40}} = \frac{{}_{2}p_{40} \times {}_{0.5}p_{42}}{{}_{0.9}p_{40}} \\ &= \frac{{}_{2}p_{40}\left(1 - 0.5q_{42}\right)}{1 - 0.9q_{40}} \quad \text{under UDD} \\ &= \frac{{}_{2}p_{40}\left(1 - 0.5q_{42}\right)}{1 - 0.9q_{40}} = \frac{{}_{2}p_{40}\left(1 - 0.5\left(1 - p_{42}\right)\right)}{1 - 0.9\left(1 - p_{40}\right)} \\ &= \frac{\frac{\ell_{42}}{\ell_{40}}\left(1 - 0.5\left(1 - \frac{\ell_{43}}{\ell_{42}}\right)\right)}{1 - 0.9\left(1 - \frac{\ell_{41}}{\ell_{40}}\right)} = \frac{\frac{9700}{10000}\left(1 - 0.5\left(1 - \frac{9400}{9700}\right)\right)}{1 - 0.9\left(1 - \frac{9900}{10000}\right)} = 0.96367 \end{aligned}$$

   then ${}_{1.6}q_{40.9} = 1 - 0.96367 = 0.03633$.

4. Under **CFM**

   (a) We can write

$$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_{2}p_{41} \times_{0.6} p_{43} = 1 - {}_{2}p_{41} \times \left(p_{43}\right)^{0.6} \\ &= 1 - \frac{\ell_{43}}{\ell_{41}} \times \left(\frac{\ell_{44}}{\ell_{43}}\right)^{0.6} = 1 - \frac{9400}{9900} \times \left(\frac{9000}{9400}\right)^{0.6} = 0.074958 \end{aligned}$$

   (b) Now, from 3. b. we have

$${}_{1.6}p_{40.9} = \frac{{}_{2}p_{40} \times {}_{0.5}p_{42}}{{}_{0.9}p_{40}} = \frac{\frac{\ell_{42}}{\ell_{40}}\left(\frac{\ell_{43}}{\ell_{42}}\right)^{0.5}}{\left(\frac{\ell_{41}}{\ell_{40}}\right)^{0.9}} = \frac{\frac{9700}{10000}\left(\frac{9400}{9700}\right)^{0.5}}{\left(\frac{9900}{10000}\right)^{0.9}} = 0.96356$$

   then ${}_{1.6}q_{40.9} = 1 - 0.96356 = 0.03644$.

**<u>Problem 2.</u> (8 marks)**

Given

$$S_0(x) = \left(1 - \frac{x}{100}\right)^{\frac{1}{2}} \quad \text{for} \quad 0 \le x \le 100$$

1. **(2 marks)** Calculate the probability that a life age 36 will die between ages 51 and 64.
2. **(3 marks)** Calculate the probability that a 40–year-old will survive to age 42 if the force of mortality is $\mu_x = kx^n$ with $k = 1/100$ and $n = 1$
3. **(3 marks)** The benefit under an $n$–year deferred whole life policy, with benefit payable at the moment of death, is twice that of a similar non–deferred whole life insurance. The expected present value for these insurances are equal and $\mu = 0.08$ and $\delta = 0.06$. Determine $n$

**<u>Solution of problem 2.</u>**

1. The probability that a life age 36 will die between ages 51 and 64 is given by

$$ {}_{15|13}q_{36} = {}_{15}p_{36} - {}_{15+13}p_{36} = S_{36}(15) - S_{36}(28) = \frac{S_0(51) - S_0(64)}{S_0(36)}. $$

   So we need to calculate the values of $S_0(36)$, $S_0(51)$ and $S_0(64)$.

$$ S_0(36) = \sqrt{0.64} = 0.8, \;\; S_0(51) = \sqrt{0.49} = 0.7 \text{ and } \;\; S_0(64) = \sqrt{0.36} = 0.6, $$

   then

$$ {}_{15|13}q_{36} = \frac{0.7 - 0.6}{0.8} = \frac{1}{8} = 0.125. $$

2. We are asked to calculate

$$ \begin{aligned} {}_2p_{40} &= \exp\left(-\int_{40}^{42} \frac{x}{100} dx\right) = \exp\left(-\int_0^2 \frac{40 + x}{100} dx\right) \\ &= \exp\left(-\frac{1}{100}\left[\frac{x^2}{2}\right]_{40}^{42}\right) = \exp\left(-\frac{1}{100}\left(\frac{42^2}{2} - \frac{40^2}{2}\right)\right) \\ &= e^{-\frac{41}{50}} = e^{-0.82} = 0.44043 \end{aligned} $$

3. Let the benefit under the deferred whole life policy be $b_d$ and the benefit under the non–deferred policy $b$. Then $b_d = 2b$,, and $b_d \; {}_{n|}\bar{A}_x = b\bar{A}_x$. It follows that ${}_{n|}\bar{A}_x = 0.5\bar{A}_x$. Therefore

$$ \begin{aligned} e^{-(\mu+\delta)n}\bar{A}_x &= 0.5\bar{A}_x \quad \text{that is} \quad e^{-(\mu+\delta)n} = \frac{1}{2} \\ n &= \frac{\ln(2)}{\mu + \delta} = \frac{\ln(2)}{0.14} = 4.951 \simeq 5 \text{ years} \end{aligned} $$

**Problem 3. (8 marks)**

1. **(2 marks)** The force of mortality is $\mu_x = \frac{1}{120-x}$ for $x < 120$. Calculate ${}_{4|5}q_{30}$.
2. **(2 marks)** Age at death is uniformly distributed on $(0, \omega]$. You are given that $q_{10} = \frac{1}{45}$. Determine $\mu_{10}$.
3. **(2 marks)** The force of mortality is given by

$$ \mu_x = \frac{1}{120 - x} + \frac{1}{160 - x} \quad \text{for} \quad 0 < x < 120 $$

   Calculate the probability that (60) will die within the next 10 years.

4. **(2 marks)** A continuous whole life insurance provides a death benefit of 1 plus a return of the net single premium with interest at $\delta = 0.04$. The net single premium for this insurance is calculated using $\mu = 0.04$ and force of interest $2\delta$. Calculate the net single premium.

**Solution of problem 3.**

1. We know that for $\mu_x = \frac{1}{\omega-x}$, $S_x(t) = {}_tp_x = 1 - \frac{t}{\omega-x}$. In our case $\omega = 120$, then

$$ {}_{4|5}q_{30} = {}_4p_{30} - {}_9p_{30} = \left(1 - \frac{4}{120 - 30}\right) - \left(1 - \frac{9}{120 - 30}\right) = \frac{5}{90} = \frac{1}{18} = 0.05556. $$

2. We have

$$ q_{10} = \frac{1}{\omega - 10} = \frac{1}{45} \text{ so } \omega = 55 \text{ and then } \mu_{10} = \frac{1}{\omega - 10} = \frac{1}{55 - 10} = \frac{1}{45} = 0.02222. $$

3. Since the force of mortality is the sum of two uniform forces, the survival probability is the product of the corresponding uniform probabilities,

$$ {}_{10}p_{60} = \frac{50}{60} \times \frac{90}{100} = \frac{3}{4} = 0.75 $$

so the answer is

$$ {}_{10}q_{60} = 1 - 0.75 = 0.25. $$

**Or in details**: we have

$$ \begin{aligned} {}_{10}p_{60} &= \exp\left(-\int_0^{10} \mu_{60+u} du\right) = \exp\left(-\int_0^{10}\left(\frac{1}{120-60-u} + \frac{1}{160-60-u}\right)du\right) \\ &= \exp\left(-\int_0^{10} \frac{1}{60-u} du\right) \exp\left(-\int_0^{10} \frac{1}{100-u} du\right) \\ &= \exp\left([\ln(60-u)]_0^{10}\right) \exp\left([\ln(100-u)]_0^{10}\right) \\ &= \exp\left(\ln\left(\frac{50}{60}\right)\right) \exp\left(\ln\left(\frac{90}{100}\right)\right) = \frac{50}{60}\frac{90}{100} = \frac{3}{4} = 0.75 \end{aligned} $$

4. Let $A$ be the net single premium. Then

$$ \begin{aligned} A &= \int_0^{\infty} \left(1 + Ae^{0.04t}\right) e^{-0.12t} 0.04 dt \\ &= \int_0^{\infty} 0.04 e^{-0.12t} dt + \int_0^{\infty} 0.04 A e^{-0.08t} dt \\ &= \frac{4}{12} \int_0^{\infty} 0.12 e^{-0.12t} dt + \frac{4}{8} A \int_0^{\infty} 0.08 e^{-0.08t} dt = \frac{1}{3} + \frac{1}{2} A \end{aligned} $$

solving we get $A = \frac{2}{3} = 0.6667$.

**Problem 4. (8 marks)** You are given:

(i) $\ddot{a}_x = 5.6$, $e_x = 8.83$ and $e_{x+1} = 8.29$

(ii) The expected present value of a 2–year certain–and–life annuity–due of 1 on $(x)$ is $\ddot{a}_{\overline{x:\overline{2|}}} = 5.6459$.

1. **(2 marks)** Calculate the effective interest rate $i$.

2. **(3 marks+1 bonus)** For a 2–year certain–and–life annuity immediate on (65), payments are $(1+0.05)^k$ at time $k$, $i = 0.05$ and the survival time is uniformly distributed with limiting age 115. Calculate the expected present value of this annuity.

3. **(3 marks+1 bonus)** A special temporary 3–year life annuity–due on (30) pays $k$ at the beginning of year $k$, $k = 1,\ 2,\ 3$. You are given: $i = 0.04$, $q_{30} = 0.10$, $q_{31} = 0.15$ and $q_{32} = 0.20$. Compute the expected present value of this annuity.

**Solution of problem 4.**

1. For a 2–year certain–and–life annuity, we can write

$$ \ddot{a}_{\overline{x:\overline{2|}}} = \ddot{a}_{\overline{2|}} + {}_{2|}\ddot{a}_x = 1 + v + \sum_{k=2}^{\infty} v^k {}_kp_x $$

For the whole life annuity we have

$$ \ddot{a}_x = 1 + vp_x + \sum_{k=2}^{\infty} v^k {}_kp_x. $$

The difference $\ddot{a}_{\overline{x:2}|} - \ddot{a}_x$ is equal to $v(1-p_x) = \frac{1}{1+i}(1-p_x) = 5.6459 - 5.6 = 0.0459$. The unknown parameters are $p_x$ and $v$ or $i$. And $p_x$ can be calculated from using the recursive equation for $e_x$ we can write

$$e_x = p_x(1+e_{x+1}) \iff p_x = \frac{e_x}{1+e_{x+1}} = \frac{883}{929} = 0.95048$$

Then we have $1+i = \frac{1-p_x}{0.0459} = \frac{1-0.950}{0.0459} = 1.089325$ that is $i = 0.089325$.

2. We have $\ddot{a}_{\overline{65:2}|} = \ddot{a}_{\overline{2}|} + {}_{2|}\ddot{a}_{65}$, but $\ddot{a}_{\overline{2}|} = 1 + (1+i)v = 2$ and since $(1+i)v = 1$, we have

$$ {}_{2|}\ddot{a}_{65} = \sum_{k=3}^{115-65-1} (1+i)^k v^k \times {}_kp_{65} = \sum_{k=3}^{49}\left(1-\frac{k}{50}\right) = \frac{564}{25} = 22.56$$

therefore $\ddot{a}_{\overline{65:2}|} = 2 + 22.56 = 24.56$

3. The APV of the annuity is given by

$$\sum_{k=0}^{2} (k+1)\, v^k \;{}_kp_{30} = 1 + \frac{0.90}{1.04}2 + \frac{0.85\times 0.90}{1.04^2}3 = 2.7308 + 2.1219 = 4.8527$$

**Problem 5. (8 marks)**. For a whole life insurance of 1 on $(x)$ with benefits payable at the moment of death, you are given:

$i)$ the force of interest at time $t$ is $\delta_t = \begin{cases} 0.02 & \text{if } t < 12 \\ 0.03 & \text{if } t \geq 12 \end{cases}$

$ii)$ the force of mortality at age $x+t$ is $\mu_{x+t} = \begin{cases} 0.04 & \text{if } t < 5 \\ 0.05 & \text{if } t \geq 5 \end{cases}$

1. **(4 marks + 1 bonus)** Calculate the actuarial present value of this insurance.

A special whole life insurance on (35) pays a benefit at the moment of death. You are given:
(i) The benefit for death in year $k$ is $9000 + 1000k$, but never more than $20,000$.
(ii) For $i = 0.06$ we are given $1000A_{35} = 128.72$, $1000{}_{10}E_{35} = 543.18$ and $1000A_{45} = 201.20$
(iii) $1000(IA)^{1}_{35:\overline{10}|} = 107.98$
(iv) Premiums are payable annually in advance.

2. **(2 marks)** Decompose this special whole life insurance into a level whole life insurance of $x$, plus a $n$–year increasing term insurance of $y$, plus a $m$–year deferred insurance of $z$. (You are asked to find $x$, $y$, $z$, $n$ and $m$)

3. **(2 marks)** Calculate the net single premium for the policy

**Solution of problem 5.**

1. $\bar{A}_x$ denotes the actuarial present value of the insurance. This insurance can be decomposed into three time intervals: $(0,5)$, $(5,12)$ and $(12,+\infty)$.

$$\begin{aligned} \bar{A}_x &= \bar{A}^{1}_{x:\overline{5}|} + {}_{5|7}\bar{A}_x + {}_{12|}\bar{A}_x \\ &= \frac{0.04}{0.04+0.02}\left(1-e^{-0.06\times 5}\right) + e^{-0.06\times 5}\frac{0.05}{0.05+0.02}\left(1-e^{-0.07\times 7}\right) + e^{-0.79}\frac{0.05}{0.05+0.03} \\ &= \frac{4}{6}\left(1-e^{-0.3}\right) + \frac{5}{7}\left(e^{-0.3}-e^{-0.79}\right) + \frac{5}{8}e^{-0.79} = 0.66142 \end{aligned}$$

**Or in details using integration**

$$\begin{aligned}\bar{A}_x &= \int_0^\infty e^{-\int_0^t \delta_u du}\mu_{x+t}\ {}_tp_x dt = \int_0^\infty e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt \\ &= \int_0^5 e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt + \int_5^{12} e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt + \int_{12}^\infty e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt \\ &= : I_1 + I_2 + I_3.\end{aligned}$$

For $I_1$ we have

$$\delta_t = \begin{cases} 0.02 & \text{if } t < 12 \\ 0.03 & \text{if } t \geq 12 \end{cases} \quad \mu_{x+t} = \begin{cases} 0.04 & \text{if } t < 5 \\ 0.05 & \text{if } t \geq 5 \end{cases}$$

$$I_1 = \int_0^5 e^{-0.02t}0.04e^{-0.04t}dt = \int_0^5 e^{-0.06t}0.04dt = \frac{4}{6}\left(1 - e^{-0.06\times 5}\right) = 0.17279.$$

For $I_2$ we have

$$\begin{aligned}I_2 &= \int_5^{12} e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt = \int_5^{12} e^{-0.02t}0.05e^{-\int_0^5 \mu_{x+u}du}e^{-\int_5^t \mu_{x+u}du}dt \\ &= \int_5^{12} e^{-0.02t}0.05e^{-0.04\times 5}e^{-0.05(t-5)}dt = \int_5^{12} e^{-0.02t}0.05e^{-0.04\times 5}e^{-0.05(t-5)}dt \\ &= 0.05e^{0.05}\int_5^{12} e^{-0.07t}dt = \frac{5}{7}e^{0.05}\left(e^{-0.07\times 5} - e^{-0.07\times 12}\right) = 0.20498\end{aligned}$$

And for $I_3$, we have

$$\begin{aligned}I_3 &= \int_{12}^\infty e^{-\int_0^t \delta_u du}\mu_{x+t}e^{-\int_0^t \mu_{x+u}du}dt = \int_{12}^\infty e^{-\int_0^{12} \delta_u du}e^{-\int_{12}^t \delta_u du}0.05e^{-\int_0^5 \mu_{x+u}du}e^{-\int_5^t \mu_{x+u}du}dt \\ &= \int_{12}^\infty e^{-0.02\times 12}e^{-0.03(t-12)}0.05e^{-0.04\times 5}e^{-0.05(t-5)}dt = 0.05e^{-0.79}\int_{12}^\infty e^{-0.08(t-12)}dt \\ &= e^{-0.79}\int_0^\infty 0.05e^{-0.08t}dt = 0.27728\end{aligned}$$

Finnally

$$\bar{A}_x =: I_1 + I_2 + I_3 = 0.17279 + 0.20498 + 0.28365 = 0.66142.$$

2. The insurance can be expressed as a level whole life insurance of 9000 on (35) plus a 10–year increasing term insurance of 1000, plus a 10–year deferred insurance of 11000. So $x = 9000$, $n = 10$, $y = 1000$, $m = 10$ and $z = 11000$)

3. Let $S_{\text{Pr}}$ be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned}S_{\text{Pr}} &= 9000A_{35} + 1000(IA)^{1}_{35:\overline{10}|} + 11000\,{}_{10}E_{35}A_{45} \\ &= 9(128.72) + 107.98 + 11(0.54318)(201.20) = 2468.63.\end{aligned}$$