Frequently Used Continues Probability Distributions

Q1) Find the moment generating function for the general normal distribution.

Q2) Show that the moment generating function of the random variable X which is Chi square distribution with degree of freedom is $M(t) = (1 - 2t)^{-v/2}$.

$x \sim \chi_v^2$

$$f(x) = \left(\frac{1}{2}\right)^{\frac{v}{2}} \frac{1}{\Gamma\frac{v}{2}} e^{-\frac{1}{2}x} x^{\frac{v}{2}-1}, x > 0$$

$$M_x(t) = E(e^{xt}) = (\frac{1}{2})^{\frac{v}{2}} \frac{1}{\Gamma\frac{v}{2}} \int_0^\infty e^{xt} e^{-\frac{1}{2}x} x^{\frac{v}{2}-1} dx$$

$$= (\frac{1}{2})^{\frac{v}{2}} \frac{1}{\Gamma\frac{v}{2}} \int_0^\infty e^{-x\frac{(1-2t)}{2}} x^{\frac{v}{2}-1}\, dx$$

$$= (\frac{1}{2})^{\frac{v}{2}} \frac{1}{\Gamma\frac{v}{2}} \frac{\Gamma\frac{v}{2}}{(\frac{(1-2t)}{2})^{\frac{v}{2}}} = \frac{1}{(1-2t)^{\frac{v}{2}}} = (1-2t)^{-\frac{v}{2}}$$

Q3) If X1 and X2 be independent r.v. that are chi-square dis. with v1 and v2 degrees of freedom, respectively.

$$X_1 \sim \chi_{v_1}^2 \ and \ \ x_2 \sim \ \chi_{v_2}^2 \ one\ independent$$

a. Show that the moment generating function of the random variable $Z = X1 + X2$ $is\ M(t) = (1 - 2t)^{-\frac{v1+v2}{2}}$

$$M_2(t) = M_{x_1+x_2}(t) = M_{x_1}(t)M_{x_2}(t) = (1-2t)^{-\frac{v_1}{2}}(1-2t)^{-\frac{v_2}{2}}$$
$$= (1-2t)^{-\frac{v_1+v_2}{2}}$$

b. What you can say about the distribution of the random variable Z .

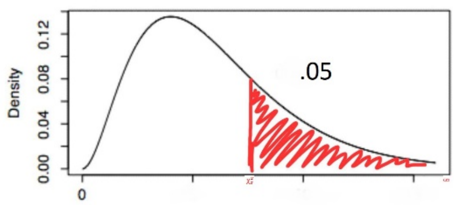
$Z \sim \chi_{v_{1+v_2}}^2$

Q4) Show that the mean and variance of gamma distribution are given by
(a) $\mu = \alpha\beta$

(b) $\sigma^2 = \alpha\beta^2$

Q5) The graph of chi-square distribution with 5 degrees of freedom is show~ below. Find the values of $1^{X2}, 2^{X2}$ for which

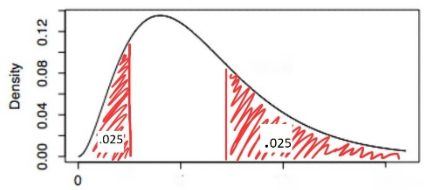
a. The shaded area on the right = 0.05,



X~ $\chi^2_{v=5}$

$P(\chi^2_v > \chi^2_2) = .05 \rightarrow 1 - p(\chi^2_v < \chi^2_2) = .05$
$\rightarrow (\chi^2_v < \chi^2_2) = .95 \rightarrow \chi^2_{v,095} = \chi^2_2 = 11.07$

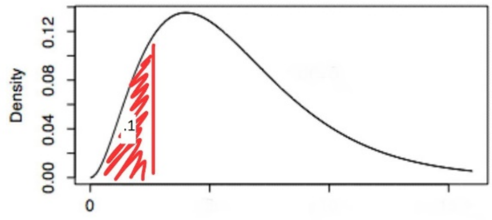
b. The total shaded area = 0.05,



$\frac{.05}{2} = .025$

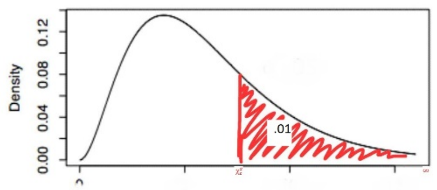
- ❖ $P(\chi^2_v < \chi^2_1) = .025 \rightarrow x^2_{v,0.025} = \chi^2_1 = .831$
- ❖ $P(\chi^2_v > \chi^2_2) = .025$
  →1-P $(\chi^2_v < \chi^2_2) = .025 \rightarrow (\chi^2_v < \chi^2_2) = .975 \rightarrow$
  $x^2_{v,0.975} = \chi^2_2 = 12.83$

c. The shaded area on the left = 0.10,



$P(\chi_v^2 < \chi_1^2) = .1 \rightarrow \chi_{v,0.1}^2 = \chi_1^2 = 1.61$

d. The shaded area one the right = 0.01 .



$P(\chi_v^2 > \chi_2^2) = .01$

$\rightarrow$1-P $(\chi_v^2 < \chi_2^2) = .01$

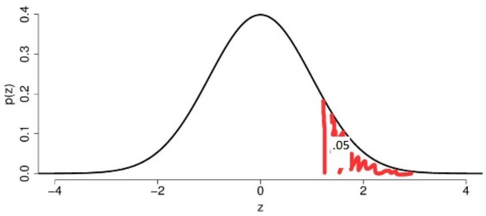
$\rightarrow$P $(\chi_v^2 < \chi_2^2) = .99$

$\chi_{v,0.99}^2 = \chi_2^2 = 15.09$

Q6) The graph of t- distribution with 9 degrees of freedom is show~ below. Find the values of t1, t2 for which

X~$T_{v=9}$

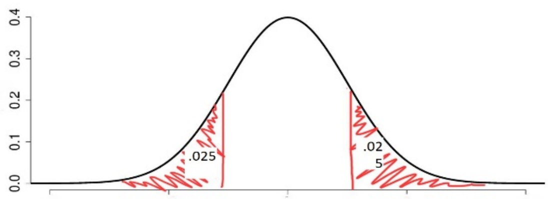
a. The shaded area one the right = 0.05,



P($T_v > t_2) = .05 \rightarrow$ 1-P($T_v < t_2) = .05$

= P($T_v < t_2) = .95 \rightarrow t_{v,.95} = t_2 = 1.833$

b. The total shaded area = 0.05,



$\frac{.05}{2} = .025$

P($T_v > t_2) = .025 \rightarrow$ 1-P($T_v < t_2) = .025 \rightarrow P(T_v < t_2) = .975$

$\rightarrow t_{v,975} = t_2 = 2.262$

$\therefore -t_2 = 2.262$

Since the distribution is symmetry

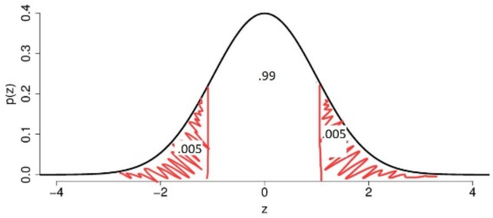
$lie\ t_1 = -t_2$

$p(T_v > t_2) = .025 \rightarrow 1 - p(T_v < t_2) = .025 \rightarrow p(T_v < t_2) =$

$.975 \rightarrow t_{v,.975} = t_2$=2.262

$\therefore t_1 = -t_2 = -2.262$

c. The total unshaded area = 0.99,



$$1-.99=.01\rightarrow\frac{.01}{2}=.005$$

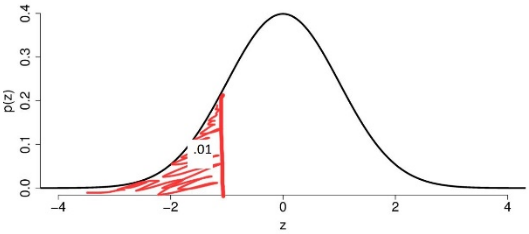
Since the distribution is symmetry

I.e $t_1=-t_2$

$p(T_v>t_2)=.005\rightarrow 1-p(T_v<t_2)=.005\rightarrow p(T_v<t_2)=$ $.995\rightarrow t_{v,.995}=t_2$=3.250

$\therefore t_1=-t_2=-3.250$

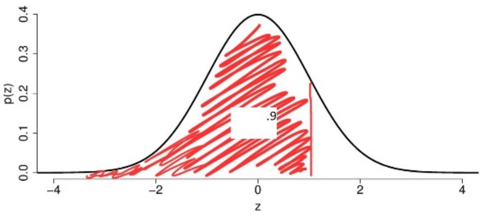
d. The shaded area on the left = 0.01,



$p(T_v<t_2)=.01\rightarrow t_1\rightarrow t_{v\,,0.01}=-t_{v,1-.01}$=$-t_{v,.99}=-2.821$

(From the symmetry of T disbstribation)

e. The area on the left of t_1 = 0.90 .



$$p(T_v < t_2) = .9 \rightarrow t_2 = t_{v,.9} = 1.383$$

Q7) Let X be an exponential random variable with parameter θ = (3). Compute the following probability: (2 ≤ X ≤ 4).

X~ exp (θ =In(3))

$$f(x) = \theta e^{-\theta x} = In(3)e^{-In(3)x}, x > 0$$

$$F(x) = 1 - e^{-\theta x} = 1 - e^{-In(3)x}, x > 0$$

$$Pdf \rightarrow P(2 \leq X \leq 4) = \int_2^4 f(x)dx = In(3)\int_2^4 e^{-In(3)x}\,dx = .0988$$

$$CDF \rightarrow P(2 \leq X \leq 4) = p(x \leq 4) - p(x \leq 2)$$
$$= 1 - e^{-In(3).4} - \left(1 - e^{-In(3)-2}\right) = .0988$$

Q8) Suppose the random variable X has an exponential distribution with parameter θ = 1.

(a) Find (x).

(b) Using (x), compute (X > 2).

X~ exp (θ =1)

$$f(x) = \theta e^{-\theta x} = e^{-x}, x > 0$$

$$F(x) = 1 - e^{-\theta x} = 1 - e^{-x}, x > 0$$

$Pdf \rightarrow P(X > 2)$

$$= \int_2^{\infty} f(x)dx = \int_2^{\infty} e^{-x}\,dx$$

$$= \frac{e^{-x}}{-1}\int_2^{\infty} -(e^{-\infty} - e^{-2}) = e^{-2}$$

$CDF \rightarrow P(X > 2) = 1 - p(x < 2) = 1 - (1 - e^{-2}) = e^{-2}$

Q9) What is the probability that a random variable X is less than its expected value, if X has an exponential distribution with parameter ?

X~ $\exp(\theta) \rightarrow f(x) = \theta e^{-\theta x}, x > 0\,, F(x) = 1 - e^{-\theta x}, x > 0, E(x) = \frac{1}{\theta}$

P(x<E(x)) = p(X<$\frac{1}{\theta}$) $= F\left(\frac{1}{\theta}\right) = 1 - e^{-\theta\frac{1}{\theta}} = 1 - e^{-1}$

Q10) Identify the distributions of the r.v. from the moment generating function:

(a) $M_x(t) = \frac{1}{1-2t} -, t < 1/2$.

$$x \sim X_v^2 \rightarrow M_x(t) = (\frac{1}{1-2t})^{\frac{v}{2}}, t < \frac{1}{2}$$

$$\therefore \frac{v}{2} = 1 \rightarrow v = 2$$

$$x \sim \exp(\theta) \rightarrow M_x(t) = \frac{\theta}{\theta - t}, t < \theta$$

$$\therefore \frac{1}{1-2t} = \frac{\frac{1}{2}}{\frac{1}{2} - t} \rightarrow \theta = \frac{1}{2}$$

(b) $M_x(t) = e^{3t+2t^2}$.

$$X \sim N(M, \sigma^2) \rightarrow M_x(t) = e^{Mt + \frac{1}{2}\sigma^2 t}$$

$$\therefore M = 3 \; and \; \frac{1}{2}\sigma^2 = 2 \rightarrow \sigma^2 = 4$$

(c) X, independent, $M_{X+Y}(t) = (\frac{2}{2-t})^3, t < \frac{1}{2}, Y \sim Exp(1/2)$.

X and Y one indep.

$$M_{x+y}(t) = (\frac{2}{2-t})^3, y \sim \exp(\theta = 2) \rightarrow M_y(t) = \frac{2}{2-t}$$

$$\therefore M_{x+y}(t) = M_x(t)M_y(t)$$

$$\rightarrow \left(\frac{2}{2-t}\right)^3 = M_x(t)\left(\frac{2}{2-t}\right)$$

$$\rightarrow M_x(t) = \frac{\left(\frac{2}{2-t}\right)^3}{\left(\frac{2}{2-t}\right)} = \left(\frac{2}{2-t}\right)^2$$

$$\therefore x \sim gamma\ (\alpha = 2\,, \beta = 2)$$

Q11) X, independent, $M_{X+Y}(t) = \frac{e2t-1}{2t-t2}$, $X \sim Exp(1/2)$, what is the distribution of Y?

X and y one indep.

$$M_{x+y}(t) = \frac{e^{2t}-1}{2t-t^2}$$

X~ exp ($\theta = 2$) $\rightarrow M_x(t) = \frac{2}{2-t}$

$$M_{x+y}(t) = M_x(t)M_y(t)$$

$\rightarrow \frac{e^{2t}-1}{2t-t^2} = \frac{2}{2-t}M_y(t)$

$$\rightarrow M_y(t) = \frac{\frac{e^{2t}-1}{t(2-t)}}{\frac{2}{2-t}} = \frac{e^{2t}-1}{t(2-t)} \cdot \frac{2-t}{2} = \frac{e^{2t}-1}{2t}$$

$=\frac{e^{2t}-e^0}{t(2-0)}$

$$\therefore X \sim uniform\ (a = 0\,, b = 2)$$