RANDOM VARIABLES, DISTRIBUTIONS AND EXPECTATIONS

DISCRETE DISTRIBUTIONS:

Q1. Consider the experiment of flipping a balanced coin three times independently.

Let X= Number of heads – Number of tails.

(a) List the elements of the sample space S.

S={TTT,TTH,THT,THH,HHH,HTT,HTH,HHT}.

(b) Assign a value x of X to each sample point.

x	TTT	TTH	THT	THH	HHH	HTT	HTH	HHT
X(x)	-3	-1	-1	1	3	-1	1	1

(c) Find the probability distribution function of X.

x	-3	-1	1	3
f(x)	1/8	3/8	3/8	1/8

(d) Find $P(X \leq 1)$ = (1/8)+(3/8)+(3/8)=7/8

(e) Find $P(X < 1)$ = (1/8)+(3/8)=1/2

(f) Find $\mu=E(X)$ = $\sum x.f(x)$ = -3(1/8)-1(3/8)+1(3/8)+3(1/8) = 0

(g) Find σ^2=Var(X) = $E(X^2) - (E(X))^2 = \sum x^2.f(x) - (E(X))^2$ = 9(1/8)+1(3/8)+1(3/8)+3(1/8)-0=3

Q2. Let X be a random variable with the following probability distribution:

x	–3	6	9
f(x)	0.1	0.5	0.4

1) Find the mean (expected value) of X, μ=E(X) = $\sum x.f(x)$ = -3(0.1)+6(0.5)+9(0.4)= 6.3

2) Find $E(X^2)$ = $\sum x^2.f(x) = (-3)^2 (0.1)+6^2(0.5)+9^2(0.4)$=51.3

3) Find the variance of X, Var (X) = σ_X^2 = $E(X^2) - (E(X))^2 = 51.3-(6.3)^2$ = 11.61

4) Find the mean of 2X+1, $E(2X+1) = \mu_{2X+1}$ = 2E(X)+E(1) = 2(6.3)+1 = 13.6

5) Find the variance of 2X+1, Var(2X+1)= $\sigma_{2X+1}^2 = 2^2$ Var(X)+Var(1)=4(11.61)+0 = 46.44

Q3. Which of the following is a probability distribution function:

(A) $f(x) = \frac{x+1}{10}$; x=0,1,2,3,4

$f(0)=(1/10) < 1$ $f(1)=(2/10) < 1$ $f(2)=(3/10) < 1$ $f(3)=(4/10) < 1$ $f(4)=(5/10) < 1$

$\sum f(x) = (1+2+3+4+5)/10 = 1.5 \neq 1$

It is not PDF

(B) $f(x) = \frac{x-1}{5}$; x=0,1,2,3,4

$f(0)=(-1/10) < 0$; It is not PDF

(C) $f(x) = \frac{1}{5}$; x=0,1,2,3,4

$f(0)=f(1)=f(2)=f(3)=f(4)=1/5$

$\sum f(x) = (1+1+1+1+1)/5 = 1$

It is PDF

(D) $f(x) = \frac{5-x^2}{6}$; x=0,1,2,3

$f(0)=(5/6) < 1$ $f(1)=(4/6) < 1$ $f(2)=(1/6) < 1$ $f(3)=(-4/6) < 0$

since $f(3)=(-4/6) < 0$, It is not PDF

Q4. Let X be a discrete random variable with the probability distribution function:
f(x) = *k*x for x=1, 2, and 3.

(i) Find the value of *k*. $\sum kx=1 \Rightarrow k + 2k + 3K = 1 \Rightarrow 6k = 1 \Rightarrow k = 1/6$

(ii) Find the cumulative distribution function (CDF), *F*(x)

$$F(x) = \begin{cases} 0 & x < 1 \\ 1\backslash 6 & 1 \le x < 2 \\ 3\backslash 6 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

(iii) Using the CDF, *F*(x), find P (0.5 < X ≤ 2.5). = (3/6)-0 = 3/6

Q5. Let X be a random variable with cumulative distribution function (CDF) given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \le x < 1 \\ 0.6 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

(a) Find the probability distribution function of X, f(x).

$$f(x) = \begin{cases} 0.25 & x = 0 \\ 0.35 & x = 1 \\ 0.40 & x = 2 \\ 0 & otherwise \end{cases}$$

(b) Find P(1≤X<2). (using both f(x) and F(x))

using f(x)

P(1≤X<2) = f(1) = 0.35

using F(x)

P(1≤X<2) = F(2-1)-F(1-1) = F(1)-F(0) = 0.6-0.25 = 0.35

(c) Find P(X>2). (using both f(x) and F(x))

using f(x)

P(X>2) = 1-P(X≤ 2) = 1-(f(1)+f(2)+f(3))= 1-(0.25+0.35+0.40)=0

using F(x)

P(X>2) = 1-P(X≤ 2) = 1-F(2) = 1-1 = 0

Q6. Consider the random variable X with the following probability distribution function:

x	0	1	2	3
f(x)	0.4	*c*	0.3	0.1

The value of *c* is 1- (0.4+0.3+0.1)

(A) 0.125 (B) 0.2 (C) 0.1 (D) 0.125 (E) − 0.2

Q7. The probability distribution for company A is given by:

x	1	2	3
f(x)	0.3	0.4	0.3

and for company B is given by:

y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that of company A.

Company A:

E(X) = 1(0.3) + 2(0.4) + 3(0.3) = 2

$E(X^2) = 1^2(0.3) + 2^2(0.4) + 3^2(0.3) = 4.6$

$Var(X) = E(X^2) – (E(X))^2 = 4.6 – 2^2 = 0.6$
Company B:
$E(Y) = 0(0.2) + 1(0.1) + 2(0.3) + 3(0.3) + 4(0.1) = 2$
$E(Y^2) = 0^2(0.2) + 1^2(0.1) + 2^2(0.3) + 3^2(0.3) + 4^2(0.1) = 5.6$
$Var(Y) = E(Y^2) – (E(Y))^2 = 5.6 – 2^2 = 0.6$
so , Var(X) is greater than Var(Y)

CONTINUOUS DISTRIBUTIONS:

Q1. If the continuous random variable X has mean μ=16 and variance σ^2=5, then $P(X=16)$ is
(A) 0.0625 (B) 0.5 (C) 0.0 (D) None of these.

Q2. Consider the probability density function:

$$f(x) = \begin{cases} k\sqrt{x} & 0 < x < 1 \\ 0 & elsewhere. \end{cases}$$

1) The value of k is: $k\int_0^1 \sqrt{x}dx = 1 \Rightarrow k\,\frac{2}{3} = 1 \Rightarrow k = \frac{3}{2}$
(A) 1 (B) 0.5 (C) 1.5 (D) 0.667
2) The probability $P(0.3 < X \leq 0.6)$ is, $1.5\int_{0.3}^{0.6} \sqrt{x}dx$
(A) 0.4647 (B) 0.3004 (C) 0.1643 (D) 0.4500
3) The expected value of X, $E(X)$ is, $1.5\int_0^1 x\sqrt{x}dx$
(A) 0.6 (B) 1.5 (C) 1 (D) 0.667
[Hint: $\int \sqrt{x}\,dx = \frac{x^{3/2}}{(3/2)} + c$]

Q3. If the cumulative distribution function of the random variable X having the form:

$$P(X \leq x) = F(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x}{x+1} & ; x \geq 0 \end{cases}$$

Then
(1) $P(0 < X < 2)$ equals to
(a) 0.555 (b) 0.333 (c) 0.667 (d) none of these.
(2) If $P(X \leq k) = 0.5$, then k equals $F(k) = 0.5 \Rightarrow \frac{k}{k+1} = 0.5 \Rightarrow 0.5k + 0.5 = k$
(a) 5 (b) 0.5 (c) 1 (d) 1.5

Q4) For each function below, determine if it can be probability density function. If so, determine c.

a. $f_1(x) = c(2x - x^3)\ ; for\ 0 < x < \frac{5}{2}$
$0 < 2x < 5$ $-5^3/2^3 < -x^3 < 0$
$-5^3/2^3 < 2x-x^3 < 5$
no value of C will satisfy $f(x) \geq 0$ for all x

b. $f_2(x) = c(2x - x^2)\ ; for\ 0 < x < \frac{5}{2}$
$0 < 2x < 5$ $-5^2/2^2 < -x^2 < 0$
$-5^2/2^2 < 2x-x^2 < 5$
no value of C will satisfy $f(x) \geq 0$ for all x

c. $f_3(x) = c(2x^2 - 4x)\ ; for\ 0 < x < 3$
$0 < 2x^2 < 5^2.$ $-10 < 4x < 0$
$-10 < 2x^2-4x < 25$

no value of C will satisfy f(x) ≥ 0 for all x

d. $f_4(x) = c(2x^2 - 4x)$; $for\ 0 < x < 2$

Q5) The r.v. X has pdf

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & otherwise \end{cases}$$

a. What is the value of c. $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow \frac{4}{3}c = 1 \Rightarrow c = \frac{3}{4}$

b. Find the following probabilities using the pdf of X:

i. $P(X < 0) = \frac{3}{4}\int_{-1}^{0}(1-x^2)dx = \frac{1}{2}$

ii. P(X≥ 1\2) $= \frac{3}{4}\int_{\frac{1}{2}}^{1}(1-x^2)dx = \frac{5}{32}$

iii. P(-1/2 < X ≤ 1/2) $= \frac{3}{4}\int_{-0.5}^{0.5}(1-x^2)dx = \frac{11}{16}$

iv. $P(X > 1)$ =0

d. What is the cdf of X. $F(x) = \begin{cases} 0 & x < -1 \\ \frac{3}{4}\int_{-1}^{x}(1-t^2)dt = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2} & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$

e. Find the probabilities in (b) using the cdf.

i. $P(X < 0) = F(0) = \frac{3}{4}(0) - \frac{1}{4}(0)^3 + \frac{1}{2} = \frac{1}{2}$

ii. P(X≥ 1\2)$= 1 - F\left(\frac{1}{2}\right) = 1 - \frac{3}{4}\left(\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{1}{2} = \frac{5}{32}$

iii. P(-1/2 < X ≤ 1/2) $= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \left(\frac{3}{4}\left(\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\right) - \left(\frac{3}{4}\left(-\frac{1}{2}\right) - \frac{1}{4}\left(-\frac{1}{2}\right)^3 + \frac{1}{2}\right) = \frac{11}{16}$

iv. $P(X > 1)$ =1-F(1)=1-1=0

Q6) Suppose continuous r.v. X has density function

$$f(x) = \begin{cases} cx^2 & 1 < x < 2 \\ 0 & otherwise \end{cases}$$

a. Find the value of the constant c. $= c\int_{1}^{2}x^2dx = 1 \Rightarrow \frac{7}{3}c = 1 \Rightarrow c = \frac{3}{7}$

b. Find P(X≥3/2) $= \frac{3}{7}\int_{\frac{3}{2}}^{2}x^2dx = \frac{37}{56}$

c. Find the cumulative distribution function of X. $F(x) = \begin{cases} 0 & x < 1 \\ \frac{3}{7}\int_{1}^{x}t^2dt = \frac{1}{7}x^3 - \frac{1}{7} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$

d. Find P(X≥3/2) using the cdf. $= 1 - F\left(\frac{3}{2}\right) = 1 - (\frac{1}{7}\left(\frac{3}{2}\right)^3 - \frac{1}{7} = \frac{37}{56})$

Q7) A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = Cxe^{-x/2} ; x > 0$$

a. What is the probability that the system functions for at least 5 months. $P(X \ge 5) = c\int_{5}^{\infty}xe^{-x/2} = 1.15c$

b. What is the probability that the system functions from 3 to 6 months.$P(3 < X < 6) = C\int_{5}^{\infty}xe^{-x/2} = 1.44c$

c. What is the probability that the system functions less than 1 month.$P(X < 1) = C\int_{0}^{1}xe^{-x/2} = 0.36c$

Q8) The cumulative distribution function of a continuous r.v. Y is given by

$$f(y) = \begin{cases} 0 & y \le 3 \\ 1 - \dfrac{9}{y^2} & y > 3 \end{cases}$$

Find

a. $P(Y \le 5)$.= F(5) = 1-(9/5^2)=16/25

b. $P(>8)$. = 1-F(8) = 1- (1-9/8^2)=9/64

c. the pdf of Y.$f(y) = \begin{cases} F'(y) = \frac{18}{y^3} & y > 3 \\ 0 & otherwise \end{cases}$

Q9) If the density function of the continuous r.v. X is $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < c \\ 0 & o.w \end{cases}$ Find

a. The value of c.$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 xdx + \int_1^c (2-x)dx = 1 \Rightarrow \frac{1}{2} + [2x - x^2]_1^c = 1 \Rightarrow -\frac{c^2}{2} + 2c - 1 = 1 \Rightarrow -\frac{c^2}{2} + 2c - 2 = 0 \Rightarrow c = 2$

b. The cumulative distribution function of X. $F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x tdt = \frac{x^2}{2} & 0 \le x < 1 \\ \int_0^1 tdt + \int_1^x (2-t)dt = 2x - \frac{x^2}{2} - 1 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$

c. $P(0.8 < X < 0.6c)$. = P(0.8 < X < 1.2) = F(1.2)-F(0.8) = (2(1.2) - $\frac{1.2^2}{2} - 1) - \frac{0.8^2}{2} = \frac{9}{25}$

EXPECTATIONS INVOLVING INDEPENDENT R.V.'S AND MOMENT GENERATING FUNCTIONS

Q1) Let X_1, X_2 and X_3 be independent r.v.'s with means 4, 9, 3 and variances 3, 7, 5 respectively. For $Y=2X_1-3X_2+4X_3$ and $Z=X_1+2X_2-X_3$, find:

a. E(Y) and E(Z).

$E(Y)=2E(X_1)-3E(X_2)+4E(X_3)=2(4)-3(9)+4(3)=-7$

$E(Z)=E(X_1)+2E(X_2)-E(X_3)=4+2(9)-3=19$

b. V(Y) and V(Z).

$V(Y)=2^2V(X_1)+(-3^2)V(X_2)+4^2V(X_3)=4(3)+9(7)+16(5)=155$

$V(Z)=V(X_1)+2^2V(X_2)+(-1)^2V(X_3)=3+4(7)+5=36$

Q2) If X and Y are independent r.v.'s with E(X)=3, E(Y)=5, V(X)=2, and V(Y)=5, find:

a. $E(XY)=E(X)E(Y)=3(5)=15$

b. $E(X^2Y)=E(X^2)E(Y)=[Var(X)+(E(X))^2]E(Y)=(2+3^2)(5)=55$

Q3) Let X and Y are independent r.v's with p.d.f $f(x)=e^{-x}; x>0$, $f(y)=e^{-y}; y>0$, find :

a. E(X) and V(X).

$E(X)=\int_0^\infty xe^{-x}dx=1$

$V(X)=E(X^2)-\left(E(X)\right)^2=\int_0^\infty x^2e^{-x}dx-\int_0^\infty xe^{-x}dx=2-1=1$

b. E(Y) and V(Y)

$E(Y)=\int_0^\infty ye^{-y}dy=1$

$V(Y)=E(Y^2)-\left(E(Y)\right)^2=\int_0^\infty y^2e^{-y}dy-\int_0^\infty ye^{-y}dy=2-1=1$

c. E(XY).

$E(X)E(Y)=(\int_0^\infty xe^{-x}d)(\int_0^\infty ye^{-y}dy)=1$

d. $E(X^2Y^3)=E(X^2)E(Y^3)=(\int_0^\infty x^2e^{-x}dx)(\int_0^\infty y^3e^{-y}dy)=2(6)=12$

Q4) Find the moment generating function of X If you know that $f(x)=2e^{-2x}, x>0$.

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}(2e^{-2x})dx=\frac{2}{-(2-t)}e^{-(2-t)x}\Big|_0^\infty=\frac{2}{2-t}; t\neq 2$$

Q5) Suppose independent r.v.'s X and Y are such that $M_{X+Y}(t)=\frac{e^{2t}-1}{2t-t^2}$. If $f(x)=\lambda e^{-\lambda x}; x>0$ what is the distribution of Y.

$$M_{X+Y}(t)=M_X(t)M_Y(t)\Rightarrow M_Y(t)=\frac{M_{X+Y}(t)}{M_X(t)}=\frac{M_{X+Y}(t)}{E(e^{tX})}=\frac{M_{X+Y}(t)}{\int_0^\infty e^{tx}(e^{-\lambda x}\lambda)}=M_{X+Y}(t)\frac{1}{\int_0^\infty e^{tx}(e^{-\lambda x}\lambda)}=\frac{e^{2t}-1}{2t-t^2}\frac{1}{\frac{\lambda}{\lambda-t}}$$

Q6) X and Y are independent and identically distributed with $M(t)=e^{3t+t^2}$. Find the mgf of Z=2X-3Y

$$M_{2X-3Y+4}(t)=M_{2X}(t)M_{-3Y+4}(t)=M_X(2t)[e^{4t}M_Y(-3t)]=e^{4t}e^{6t+4t^2}e^{-9t+9t^2}=e^{t+13t^2}$$

$$M_{2X-3Y+4}(t)=E\left(e^{t(2X-3Y+4)}\right)=E\left(e^{2tX}e^{-3tY}e^{4t}\right)=e^{4t}E\left(e^{2tX}e^{-3tY}\right)=e^{4t}E\left(e^{2tX}\right)E\left(e^{-3tY}\right)=e^{4t}M_X(2t)M_Y(-3Y)$$
$$=e^{t+13t^2}$$

Q7) Suppose X has $M_X(t)=e^{3t+t^2}$. Find the mgf of $Z=\frac{1}{4}(X-3)$and use it to find the mean and use it to find the mean and variance od Z.

$$M_Z(t)=M_{\frac{1}{4}(X-3)t}=M_{\frac{1}{4}X-\frac{3}{4}}(t)=e^{\frac{-3}{4}t}M_X\left(\frac{1}{4}t\right)=e^{\frac{-3}{4}t}e^{3\left(\frac{1}{4}t\right)+\left(\frac{1}{4}t\right)^2}=e^{\frac{-3}{4}t}e^{\frac{3}{4}t+\frac{1}{16}t^2}=e^{\frac{1}{16}t^2}$$

$E(Z) = M'_Z(t)\Big|_{t=0} = \frac{1}{16} 2te^{\frac{1}{16}t^2}\Big|_{t=0} = 0$

$E(Z^2) = M''_Z(t) = \Big|_{t=0} = \frac{2}{16}\left(t\frac{2}{16}te^{\frac{1}{16}t^2} + e^{\frac{1}{16}t^2}\right)\Big|_{t=0} = \frac{1}{8}$

Var(Z) = E (Z^2)-(E (Z))^2 = $\frac{1}{8}$

Q8) Suppose X is a r.v. for which the mgf is $M_X(t) = \frac{1}{4}(3e^t + e^{-t}), -\infty < t < \infty$.

a. Find the mean and variance of X.

$E(X) = M'_X(t)\Big|_{t=0} = \frac{1}{4}(3e^t - e^{-t})\Big|_{t=0} = \frac{1}{4}(3-1) = \frac{1}{2}$

$E(X^2) = M''_X(t)\Big|_{t=0} = \frac{1}{4}(3e^t + e^{-t})\Big|_{t=0} = \frac{1}{4}(3+1) = 1$

Var(X)=E(X^2)-(E (X))^2= 1-$(1/2)^2$=3/4

Q9) Let $f(x) = 1$; $0 \leq x \leq 1$. Use the moment generating function technique to find the moment generating function of Y=aX+b where a and b are constant.

$M_X(t) = \int_0^1 e^{tX}\,dx = \frac{e^t - 1}{t}$

$M_Y(t) = e^{bt}M_X(at) = e^{bt}\left(\frac{e^{bt} - 1}{at}\right)$

Q10) Let $f(x) = e^{-x}$; $x > 0$, find the mgf of Z=3-2X.

$M_X(t) = \int_0^\infty e^{tX}e^{-X}dx = \frac{1}{1-t}$

$M_Z(t) = e^{3t}M_X(-2t) = \frac{e^{3t}}{1+2t}$