① $f(x,y) = cxy \quad , x = 1,2,3 \; ; y = 1,2,3$

a) $\sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) = 1$

or

$c + 2c + 3c + 2c + 4c + 6c + 3c + 6c + 9c = 1 \Rightarrow c = \frac{1}{36}$

$6c + 12c + 18c = 1 \Rightarrow c = \frac{1}{36}$

$y \backslash x$	1	2	3	$f_Y(y)$
1	1c	2c	3c	6c
2	2c	4c	6c	12c
3	3c	6c	9c	18c
$f_X(x)$	6c	12c	18c	1 = total

c) $p(1 \le X \le 2, Y \le 2) \quad , x = 1,2 ; \; y = 1,2$

$\therefore p(1 \le X \le 2, Y \le 2) = p(X=1, Y=1) + p(X=1, Y=2)$

$+ p(X=2, Y=1) + p(X=2, Y=2)$

$= f(1,1) + f(1,2) + f(2,1) + f(2,2)$

$= 1c + 2c + 2c + 4c = 9c = 9 \frac{1}{36}$

b) $p(X=2, Y=3) = f(2,3) = 6c = \frac{6}{36} = \frac{1}{6}$

d) $p(X \ge 2) \; , \; x = 2, 3$

$\therefore p(X \ge 2) = p(X=2) + p(X=3) = f_X(2) + f_X(3) = 12c + 18c = 30c = \frac{30}{36}$

e) $p(Y < 2) \quad , y = 1$

$\therefore p(Y < 2) = p(Y=1) = f_Y(1) = 6c = \frac{6}{36} = \frac{1}{6}$

f) $p(X=1) = f_X(1) = 6c = \frac{6}{36} = \frac{1}{6}$

[redacted]

g) $p(Y=3) = f_Y(3) = 18c = 18/36$

①→② marginal dis. of X :

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_X(x) = p(X=x)$	6/36	12/36	18/36	1

marginal dis. of Y :

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_Y(y) = p(Y=y)$	6/36	12/36	18/36	1

are X and Y indep. ?

if $f(x,y) = f_X(x) \, f_Y(y) \quad \forall \, x = 1,2,3 \, , y = 1,2,3$

then X and Y are indep.

but if there some values of x and y which make that $f(x,y) \neq f_X(x) \, f_Y(y)$

then X and Y are not indep.

in this example, we have :

$f(1,1) = f_X(1) \, f_Y(1)$

$f(1,2) = f_X(1) \, f_Y(2)$

$\vdots$

$f(3,3) = f_X(3) \, f_Y(3)$

so as $f(x,y) = f_X(x) \, f_Y(y) \quad \forall \, x, y$, then X and Y are indep.

Q ⑤ dis. of $X|Y$: $f_{X|Y=y}(x) = \dfrac{f(x,y)}{f_y(y)}$ — $x = 1, 2, 3$

1) if $Y=1$: $f_{X|Y=1}(x) = \dfrac{f(x,1)}{f_y(1)} = \dfrac{f(x,1)}{6/36}$ $\Rightarrow$

x	1	2	3	total
$f_{X\|Y=1}(x)$	$\frac{f(1,1)}{6/36} = 1/6$	$\frac{f(2,1)}{6/36} = 1/3$	$\frac{f(3,1)}{6/36} = 1/2$	1

2) if $Y=2$: $f_{X|Y=2}(x) = \dfrac{f(x,2)}{f_y(2)} = \dfrac{f(x,2)}{12/36}$ $\Rightarrow$

x	1	2	3	total
$f_{X\|Y=2}(x)$	$\frac{f(1,2)}{12/36} = 2/12$	$\frac{f(2,2)}{12/36} = 4/12$	$\frac{f(3,2)}{12/36} = 6/12$	1

3) if $Y=3$: $f_{X|Y=3}(x) = \dfrac{f(x,3)}{f_y(3)} = \dfrac{f(x,3)}{18/36}$ $\Rightarrow$

x	1	2	3	total
$f_{X\|Y=3}(x)$	$\frac{f(1,3)}{18/36} = 3/18$	$\frac{f(2,3)}{18/36} = 6/18$	$\frac{f(3,3)}{18/36} = 9/18$	1

dis. of $Y|X$: $f_{Y|X=x}(y) = \dfrac{f(x,y)}{f_X(x)}$, $y = 1, 2, 3$

1) if $X=1$: $f_{Y|X=1}(y) = \dfrac{f(1,y)}{f_X(1)} = \dfrac{f(1,y)}{6/36}$ $\Rightarrow$

y	1	2	3	total
$f_{Y\|X=1}(y)$	$\frac{f(1,1)}{6/36} = 1/6$	$\frac{f(1,2)}{6/36} = 2/6$	$\frac{f(1,3)}{6/36} = 3/6$	1

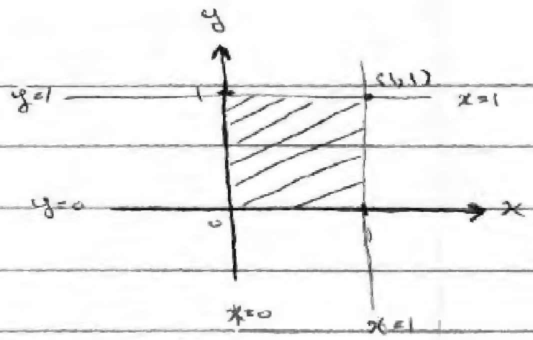
2) if $X=2$: $f_{Y|X=2}(y) = \dfrac{f(2,y)}{f_X(2)} = \dfrac{f(2,y)}{12/36}$ $\Rightarrow$

y	1	2	3	total
$f_{Y\|X=2}(y)$	$\frac{f(2,1)}{12/36} = 2/12$	$\frac{f(2,2)}{12/36} = 4/12$	$\frac{f(2,3)}{12/36} = 6/12$	1

3) if $X=3$: $f_{Y|X=3}(y) = \dfrac{f(3,y)}{f_X(3)} = \dfrac{f(3,y)}{18/36}$ $\Rightarrow$

y	1	2	3	total
$f_{Y\|X=3}(y)$	$\frac{f(3,1)}{18/36} = 3/18$	$\frac{f(3,2)}{18/36} = 6/18$	$\frac{f(3,3)}{18/36} = 9/18$	1

(3) $f(x,y) = c(x^2+y^2)\ ,\quad 0<x<1\ ,\quad 0<y<1$

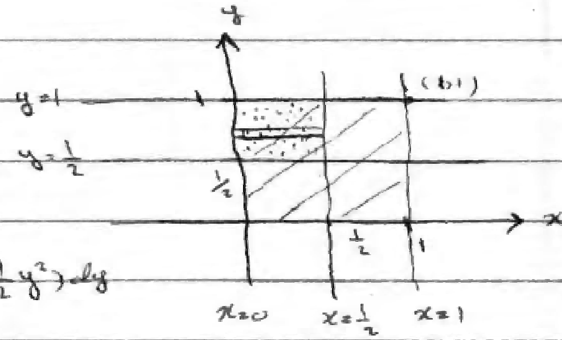


a) $\int_0^1\int_0^1 c(x^2+y^2)\,dx\,dy = 1$

$\Rightarrow c\int_0^1\left[\int_0^1 (x^2+y^2)\,dx\right]dy = 1 \Rightarrow c\int_0^1\left[\frac{x^3}{3}+xy^2\right]_0^1 dy = 1$

$\Rightarrow c\int_0^1\left(\frac{1}{3}+y^2\right)dy = 1 \Rightarrow c\left[\frac{1}{3}y+\frac{y^3}{3}\right]_0^1 = 1 \Rightarrow c\left[\frac{1}{3}+\frac{1}{3}\right]=1 \Rightarrow c=\frac{3}{2}$

b) $P(X<\frac{1}{2}, Y>\frac{1}{2})$



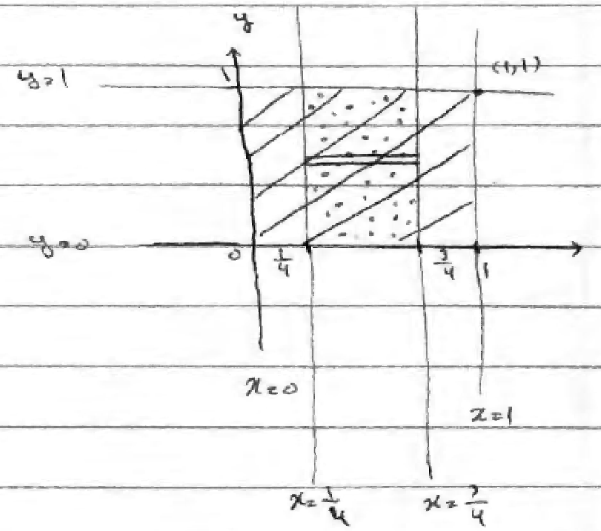
$= \frac{3}{2}\int_{1/2}^1\left[\int_0^{1/2}(x^2+y^2)\,dx\right]dy$

$= \frac{3}{2}\int_{1/2}^1\left[\frac{x^3}{3}+xy^2\right]_0^{1/2}dy = \frac{3}{2}\int_{1/2}^1\left(\frac{1}{24}+\frac{1}{2}y^2\right)dy$

$= \frac{3}{2}\left[\frac{y}{24}+\frac{y^3}{6}\right]_{1/2}^1 = \frac{3}{2}\left[\left(\frac{1}{24}+\frac{1}{6}\right)-\left(\frac{1}{48}+\frac{1}{48}\right)\right] = \frac{1}{4}$

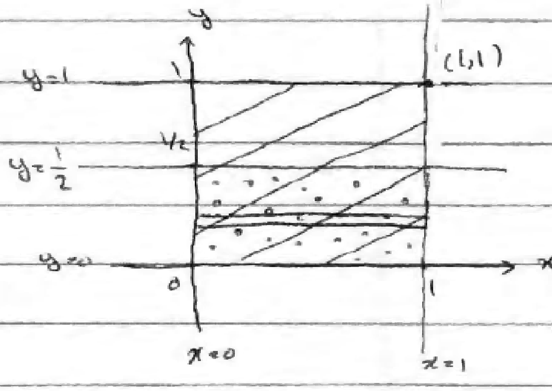
[redacted]

c) $P(\frac{1}{4}<X<\frac{3}{4})$



$= \frac{3}{2}\int_0^1\left[\int_{1/4}^{3/4}(x^2+y^2)\,dx\right]dy = \frac{39}{128}$

d) $P(Y<\frac{1}{2}) = \frac{3}{2}\int_0^{1/2}\left[\int_0^1(x^2+y^2)\,dx\right]dy$



$= \frac{3}{2}\int_0^{1/2}\left[\frac{x^3}{3}+xy^2\right]_0^1 dy$

$= \frac{3}{2}\int_0^{1/2}\left(\frac{1}{3}+y^2\right)dy = \frac{3}{2}\left[\frac{1}{3}y+\frac{y^3}{3}\right]_0^{1/2}$

$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2^3}\right] = \frac{5}{16}$

e) لكي يكون X و Y مستقلان لابد من تحقق شرطين:

1) الدالة المشتركة $f(x,y)$ لابد أن تكون عبارة عن حاصل ضرب دالتين وهما $g(x)$ وهي دالة في x و $h(y)$ وهي دالة في y

2) مجال x لا يعتمد على y وكذلك مجال y لا يعتمد على x

in this example, we can see that

$f(x,y) \neq g(x)\,h(y)$ $\therefore$ X and Y are not indep.

③ ٧ج ④ marginal dis. of X:

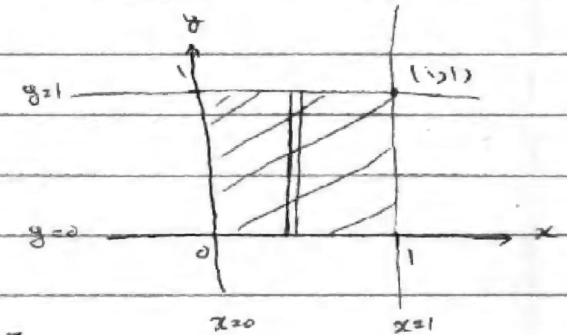
$$f_X(x) = \int_0^1 f(x,y)\,dy$$

$$= \frac{3}{2}\int_0^1 (x^2+y^2)\,dy$$

$$= \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left[x^2 + \frac{1}{3}\right]$$

$$= \frac{3}{2}x^2 + \frac{1}{2}$$

$$\therefore f_X(x) = \frac{3}{2}x^2 + \frac{1}{2}, \quad 0<x<1$$

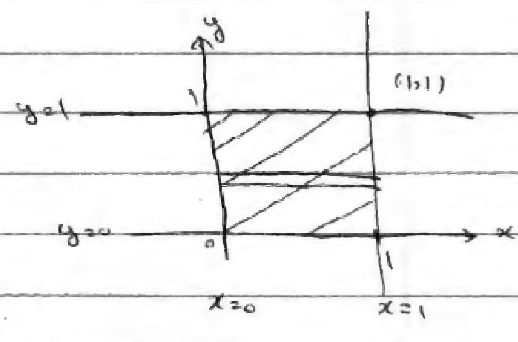


marginal dis. of Y:

$$f_Y(y) = \int_0^1 f(x,y)\,dx$$

$$= \frac{3}{2}\int_0^1 (x^2+y^2)\,dx = \frac{3}{2}y^2 + \frac{1}{2}$$

$$\therefore f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2}, \quad 0<y<1$$

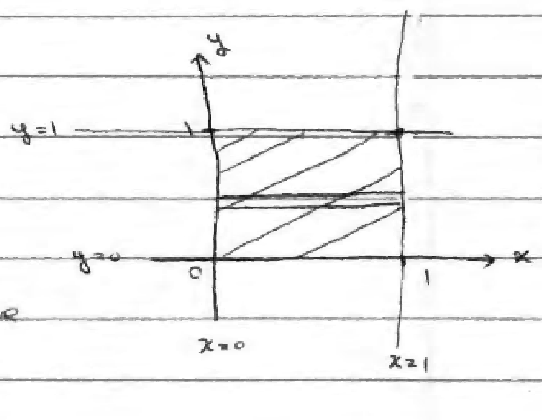


③ ٧ج ⑦ conditional dis. X|Y:

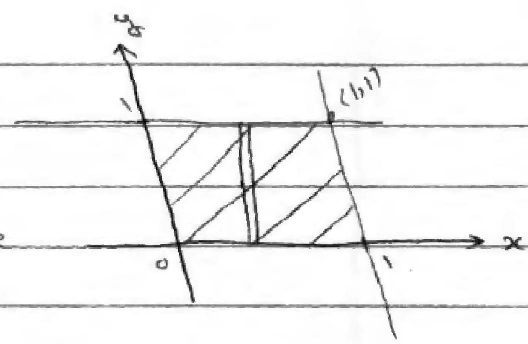
$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2+\frac{1}{2}}$$

$$= \frac{\cancel{\frac{3}{2}}(x^2+y^2)}{\cancel{\frac{3}{2}}(y^2+\frac{1}{3})} = \frac{x^2+y^2}{y^2+\frac{1}{3}}$$

for $0<x<1$ where $0<y<1$ fixed value



conditional dis. of $Y|X$:



$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{x^2+y^2}{x^2+\frac{1}{3}}$$

For $0<y<1$ where $0<x<1$ fixed value

$\boxed{Q_6}$ $f(x,y) = x+y$ $\quad 0 \le x \le 1$

$0 \le y \le 1$

marginal of x, y

$$f(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big]_0^1 = x + \frac{1}{2}$$

كذلك

$$f(y) = y + \frac{1}{2}$$

Conditional:

$$f(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{2(x+y)}{2y+1}$$

$$f(y/x) = \frac{f(x,y)}{\text{[redacted]}} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{2x+1}$$

[redacted]

$\boxed{Q_8}$ $f(x,y) = e^{-(x+y)}$ $\quad x \geq 0,\ y \geq 0$

$$f(x) = \int_0^\infty e^{-x} e^{-y}\, dy = e^{-x}\left(-e^{-y}\right]_0^\infty$$

$$= e^{-x}[1-0] = e^{-x}$$

بالمثل

$$f(y) = \int_0^\infty e^{-x} e^{-y}\, dx = e^{-y}$$

$\boxed{f(x) = e^{-x}}$ marginal of x $\qquad \boxed{f(y) = e^{-y}}$ marginal of y

Conditional x given y

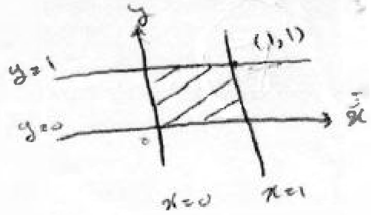
[redacted]

Conditional y given x

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{e^{-(x+y)}}{e^{-x}} = e^{-y}$$

11

$f(x,y) = x+y \qquad 0 \le x \le 1, \quad 0 \le y \le 1$



$$f(x) = \int_0^1 x+y \, dy = xy + \frac{y^2}{2}\Big]_0^1 = \boxed{x + \frac{1}{2}}, \quad 0<x<1$$

وبالمثل

$$f(y) = \int_0^1 (x+y) \, dx = \boxed{y + \frac{1}{2}}, \quad 0<y<1$$

---

a,b)

$$E(x) = \int_0^1 x f(x) dx = \int_0^1 x(x+\tfrac{1}{2}) dx$$

$\boxed{\text{or } E(X) = \int_0^1\int_0^1 x f(x,y) dx dy}$

$$= \int_0^1 x^2 + \frac{x}{2} dx = \frac{x^3}{3} + \frac{x^2}{4}\Big]_0^1 = \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$$

وبالمثل

$$E(y) = \int_0^1 y f(y) dy = \boxed{\frac{7}{12}}$$

$$E(x^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^2(x+\tfrac{1}{2}) dx$$

$\boxed{\text{or } E(x^2) = \int_0^1\int_0^1 x^2 f(x,y) dx dy}$

$$= \int_0^1 x^3 + \frac{x^2}{2} dx = \frac{x^4}{4} + \frac{x^3}{6}\Big]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \boxed{\frac{5}{12}}$$

بالمثل $E(y^2) = \int_0^1 y^2 f(y) dy = \boxed{\frac{5}{12}}$

$$V(x) = E(x^2) - [E(x)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

بالمثل $V(y) = E(y^2) - [E(y)]^2 = \frac{11}{144}$

c, d)

$$\sigma_x = \sqrt{Var(x)} = \sqrt{\frac{11}{144}} = 0.2764$$

بالمثل

$$\sigma_y = \sqrt{Var(y)} = \sqrt{\frac{11}{144}} = 0.2764$$

---

e)

$$Cov(x,y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_0^1\int_0^1 xy\,(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^2y + xy^2\,dx\,dy = \int_0^1 \left.\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2\,dy = \left.\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

$$Cov(x,y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$$

---

f)

$$\rho = \frac{Cov(x,y)}{\sqrt{Var(x)Var(y)}} = \frac{-1/144}{\sqrt{11/144 \times 11/144}} = \frac{-1/\cancel{144}}{11/\cancel{144}} = -\frac{1}{11}$$

$$= \pm 0.091$$

relation weak negative.

$\boxed{4}$ 13

$E(x)=2,\ E(y)=3,\ E(XY)=10$

$E(x^2)=10,\ E(y^2)=16$

a) $Var(x) = E(x^2) - [E(X)]^2$

$= 10 - 2^2 = \boxed{6}$

$Var(y) = E(y^2) - [E(y)]^2$

$= 16 - 3^2 = 7$

$Cov(x,y) = E(xy) - E(x)E(y)$

$= 10 - [2 \times 3] = 4$

b) $\rho = \quad Cov(x,y) \quad = \quad 4 \quad = \quad 4 \quad =$

[redacted]

$\sqrt{var(x)Var(y)} \qquad \sqrt{6 \times 7}$

$= 0.6172$

$\rho = -\frac{1}{4}$, $Var(x) = 3$

$Var(y) = 5$

$Cov(x, y) = ??$

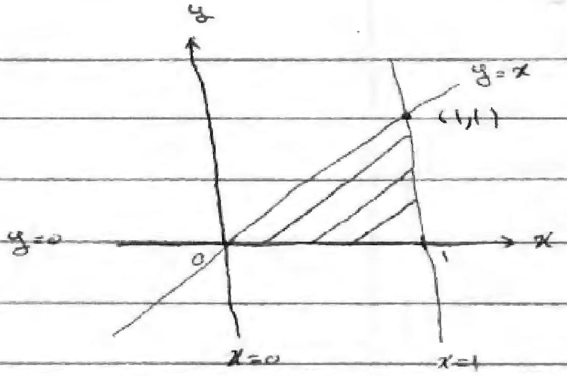
$$\rho = \frac{Cov(x,y)}{\sqrt{Var(x)\,Var(y)}}$$

$$-\frac{1}{4} = \frac{Cov(x,y)}{\sqrt{3 \times 5}}$$

[redacted]

Q18

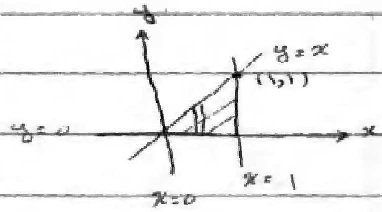
$f(x,y) = 8xy \quad , \ 0<x<1 \quad , \ 0<y<x$



a) $f_X(x) = \int_0^x f(x,y)\,dy = 8x \int_0^x y\,dy$

$= 8x \frac{y^2}{2}\Big|_0^x = 4x\left[x^2\right]$

$= 4x^3$ for $0<x<1$

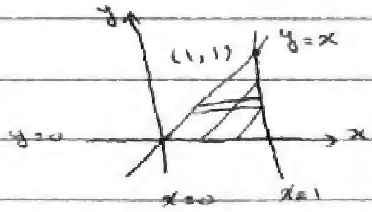


b) $f_Y(y) = \int_y^1 f(x,y)\,dx$

$= 8y \int_y^1 x\,dx = 8y \frac{x^2}{2}\Big|_y^1$

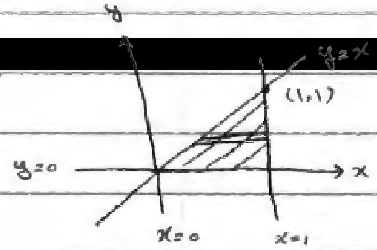
$= 4y\left[1-y^2\right] = 4y - 4y^3$

$= 4(y-y^3)$ for $0<y<1$



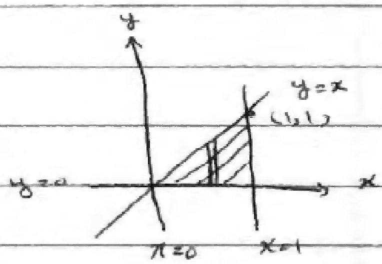
c) [redacted]

for $y<x<1$ where $0<y<1$ fixed value



d) $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}$

for $0<y<x$ where $0<x<1$ fixed value.



Q19

$E(X|Y=y) = \int_y^1 x f_{X|Y=y}(x)\,dx = \frac{2}{1-y^2} \int_y^1 x^2\,dx$

$= \frac{2}{1-y^2} \frac{x^3}{3}\Big|_y^1 = \frac{2}{3} \frac{1}{1-y^2}(1-y^3) = \frac{2}{3} \frac{(1-y^3)}{(1-y^2)}$

$E(Y|X=x) = \int_0^x y f_{Y|X=x}(y)\,dy = \frac{2}{x^2}\int_0^x y^2\,dy = \frac{2}{x^2}\frac{y^3}{3}\Big|_0^x$

$= \frac{2}{3}\frac{1}{x^2}\left[x^3\right] = \frac{2}{3}x$

Q20

$Var(Y|X) = E(Y^2|X) - E(Y|X)^2$

$E(Y^2|X) = \int_0^x y^2 f_{Y|X=x}(y)\, dy = \frac{2}{x^2}\int_0^x y^3\, dy$

$= \frac{2}{x^2}\, \frac{y^4}{4}\Big|_0^x = \frac{2}{4}\, \frac{1}{x^2}\left[x^4\right] = \frac{1}{2}x^2$

$\therefore Var(Y|X) = \frac{1}{2}x^2 - \left[\frac{2}{3}x\right]^2 = \left[\frac{1}{2} - \frac{2^2}{3^2}\right]x^2$

$= \frac{1}{18}x^2$