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Solution First Midterm Exam Math 580 (Theory Measure I)

Exercise 1:(4 points)

By definition, $\limsup B_n = \bigcap_n \left(\bigcup_{k\geq n} B_k\right)$ and $\liminf B_n = \bigcup_n \left(\bigcap_{k\geq n} B_k\right)$. As

$$\begin{aligned}\bigcup_{k\geq n} B_k &= \left(\bigcup_{k\geq n, k \text{ even}} B_k\right) \cup \left(\bigcup_{k\geq n, k \text{ odd}} B_k\right)\\ &= \left[-1; 2+\frac{1}{n^2}\right[\cup \left]-2-\frac{1}{n}; 1\right]\\ &= \left]-2-\frac{1}{n}; 2+\frac{1}{n^2}\right[.\end{aligned}$$

Then $\bigcap_{n\geq 1} \left]-2-\frac{1}{n}; 2+\frac{1}{n^2}\right[= [-2, 2]$. So $\limsup B_n = [-2; 2]$. Similarly,

$$\begin{aligned}\bigcap_{k\geq n} B_k &= \left(\bigcap_{k\geq n, k \text{ even}} B_k\right) \cap \left(\bigcap_{k\geq n, k \text{ odd}} B_k\right)\\ &= [-1; 2] \cap [-2; 1]\\ &= [-1, 1]\end{aligned}$$

Then $\liminf B_n = [-1; 1]$.

Exercise 2:(4 points)

$\mathcal{F} = \{A = [0; 2], B = [1, 3]\}$. We have: $A \cap B = [1; 2]$, $A \cup B = [0; 3]$, $B \setminus A =]2; 3]$, $A \setminus B = [0; 1[$. $A \cap B$,$A \setminus B$, $B \setminus A$ and $(A \cup B)^c = [0; 3]^c$ form a partition of $\mathbb{R}$. Then the $\sigma-$algebra $\mathcal{A}$ generated by $\mathcal{F}$ has $2^4 = 16$ elements. So

$$\begin{aligned}\mathcal{A} = &\{\emptyset, \mathbb{R}, A, A^c, B, B^c, A\cup B, (A\cup B)^c, A\cap B, (A\cap B)^c, A\setminus B, (A\setminus B)^c,\\ &B\setminus A, (B\setminus A)^c, A\Delta B, (A\Delta B)^c\}.\end{aligned}$$

Exercise 3:(5 points)

• $\mu(\emptyset) = 0$ because $\emptyset$ is a countable set.

• Let $A, B \in \mathcal{A}$ such that $A \subset B$.
- If B is a countable set then also A is countable. It follows

$$0 = \mu(A) \leq \mu(B) = 0.$$

- If B is uncountable then $\mu(B) = 1$. As $\mu(A) \leq 1$ then $\mu(A) \leq \mu(B)$.
• Let $(A_n)_{n\geq 1}$ be a disjoint sequence of elements of $\mathcal{A}$.
- If for all $n \geq 1$, A_n is a countable set, then $A := \cup_n A_n$ is also countable. We have: $\mu(A) = 0$ and $\mu(A_n) = 0$, $\forall n$. So $\mu(\cup_n A_n) = \sum_n \mu(A_n)$.
- If there exists $n_0 \geq 1$ such A_{n_0} is uncountable then $A = \cup_n A_n$ is also uncountable. As $A_{n_0} \in \mathcal{A}$ then ${A_{n_0}}^c$ is countable set by definition of $\mathcal{A}$. So $\forall n \neq n_0$, $A_n \subset {A_{n_0}}^c$. As A_n is countable for every $n \neq n_0$ then $\mu(\cup_n A_n) = 1$, $\sum_{n\neq n_0} \mu(A_n) = 0$ and $\mu(A_{n_0}) = 1$. Hence we get: $\mu(\cup_n A_n) = \sum_n \mu(A_n)$. We deduce that μ is a measure on $\mathcal{A}$.

Exercise 4:(7 points)

1. As $\mathcal{A}$ is a $\sigma-$algebra then for all $n \geq 1$, $B_n := \bigcap_{k\geq n} A_k \in \mathcal{A}$. But for every $k \geq n$, $B_n \subset A_k$. By monotonicity of the measure m on $\mathcal{A}$, we have: for $k \geq n$, $m(B_n) \leq m(A_k)$. It follows that

$$m(B_n) \leq \inf_{k\geq n} m(A_k).$$

2. For every $n \geq 1$, we have: $B_n = A_n \cap B_{n+1}$. So $B_n \subset B_{n+1}$. Also

$$b_n := \inf_{k\geq n} m(A_k) = \min\{m(A_n), b_{n+1}\} \leq b_{n+1}.$$

3. $\bigcup_{n\geq 1} \left(\bigcap_{k\geq n} A_k\right) = \bigcup_n B_n$ is the union of an increasing sequence of elements of $\mathcal{A}$. By the continuity of the measure, we get:

$$m(\cup_{n\geq 1} B_n) = \lim_{n\to\infty} m(B_n) \leq \lim_{n\to\infty} b_n = \sup_{n\geq 1} \inf_{k\geq n} m(A_k).$$

4. By definition of liminf, $\liminf_n m(A_n) = \sup_n \inf_{k\geq n} m(A_k)$, we deduce from 3.:

$$m(\liminf_n A_n) \leq \liminf_n m(A_n).$$