إعداد: د. محفوظ عالم

King Saud University
Department of Mathematics
M-203[Final Examination] (Differential and Integral Calculus)
(II-Semester 1430/31)

Time: 3 hrs Max.Marks:50

Marking Scheme: Q.No:1[3+4+4], Q.No2:[4+5], Q.No:3[5+5], Q.No:4[4+3], Q.No5:[5+8]

Q.No: 1 (a) Determine whether the sequence $\left\{(e^n+1)^{1/n}\right\}$ converges or diverges and if it converges , find its limit.

(b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+4}$ is absolutely convergent, conditionally convergent, or divergent.

(c) Find the **interval of convergence** and **radius** of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n4^n}$.

Q.No: 2 (a) Give the **Maclaurin's series** for the function $f(x)=\cos x$ and hence approximate the Integral $\int_0^1 x\cos(x^3)dx$ up to **four decimal** places by using the **first three** non-zero terms.

(b) Evaluate the integral $\int_0^1\int_x^{\sqrt{x}} e^{x/y}dydx$

Q.No: 3 (a) Find the **surface area** of the portion of the hemi-sphere $z=\sqrt{25-x^2-y^2}$ that lies above the region R bounded by the circle $x^2+y^2=9$.

(b) Evaluate the integral by changing to **spherical coordinates:**

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dzdxdy .$$

Q.No: 4 (a) Show that the line integral $\int_{(\pi/2,0)}^{(\pi,1)}(e^{3y}-y^2\sin x)dx+(3xe^{3y}+2y\cos x)dy$ is **independent of path** by finding the **potential function** and hence find its value .

(b) Use **Green's theorem** to evaluate the line integral $\oint_C (x^4+4)dx+xydy$, where C is the cardioid $r=1+\cos\theta$.

Q.No: 5 (a) Use **Divergence theorem** to evaluate the **flux** of the vector field $\overline{F}(x,y,z)=x^2\bar{i}+(y^2-2xy)\bar{j}+(4z-2yz)\bar{k}$ through the surface S bounded by the **cone** $z=\sqrt{y^2+x^2}$ and the **plane** $z=9$.

(b) **Verify Stokes's** theorem for the vector field $\overline{F}(x,y,z)=(3z+1)\bar{i}+(x^2-1)\bar{j}+(y^3+x)\bar{k}$ and the surface S, where S is the paraboloid $z=9-x^2-y^2$ cut off by the xy-plane.

Final Exam

$\boxed{M-203}$ P①

$\boxed{\text{Second Semester (1430/31)}}$ P①

#Q:1 (a) [Marks: 3]

$$y = (e^x+1)^{1/x}$$

$$\ln y = \frac{\ln(e^x+1)}{x}$$

$$\underset{x\to\infty}{Lt} \ln y = \underset{x\to\infty}{Lt} \frac{\ln(e^x+1)}{x} = \underset{x\to\infty}{Lt} \frac{\frac{e^x}{e^x+1}}{1}$$

$$= \underset{x\to\infty}{Lt} \frac{e^x}{e^x} = 1$$

$$\underset{x\to\infty}{Lt} \ln y = 1 \implies \underset{x\to\infty}{Lt}\, y = e^1 = e$$

#1(b) $\sum_{n=1}^{\infty} |U_{n+1}| = \sum_{n=1}^{\infty} \frac{n}{n^2+4} = \sum a_n$ [Marks: 4]

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n} = \sum b_n$ d'gt

$$\underset{n\to\infty}{Lt} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \underset{n\to\infty}{Lt} \frac{n^2}{n^2+4} = 1 \neq 0$$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$ is not absolutely c'gt

AST $\quad a_n = \frac{n}{n^2+4}$

(i) $\underset{n\to\infty}{Lt}\, a_n = 0$

(ii) DBC $\quad f(x) = \frac{x}{x^2+4}$

$$f'(x) = \frac{(1)(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2} \leq 0$$

for higher x

$\Rightarrow$ C.C.

P (2)

#1 (c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)\,4^{n+1}} \times \frac{n\,4^n}{x^n}\right|$ [Marks: 4]

$$= \frac{n}{4(n+1)}\,|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{4}|x|$$

c'gt for $\frac{1}{4}|x| < 1 \Rightarrow |x| < 4$

$$-4 < x < 4$$

Check c'gence at $x = -4$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

d'gt

Check c'gence at $x = 4$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\,4^n}{n\,4^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$ c'gt

Interval of c'gence $(-4, 4]$

$$r = \frac{4-(-4)}{2} = 4$$

P (3)

#Q: 2 (a) [Marks: 4]

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!} + \cdots$$

$$\Rightarrow \cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots$$

$$x\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \cdots$$

$$\int_0^1 x\cos(x^3)\,dx \approx \int_0^1 \left[x - \frac{x^7}{2!} + \frac{x^{13}}{4!}\right] dx$$

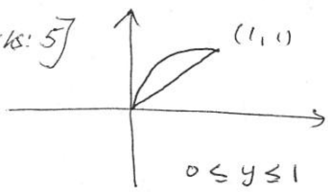
$$= \left[\frac{x^2}{2} - \frac{x^8}{16} + \frac{x^{14}}{336}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{16} + \frac{1}{336}$$

$$= \frac{168 - 21 + 1}{336} = \frac{148}{336}$$

$$\approx 0.4405$$

#2 (b) $\int_0^1 \int_x^{\sqrt{x}} e^{x/y}\,dy\,dx$ [Marks: 5]



$0 \le y \le 1$

$y^2 \le x \le y$

$$= \int_0^1 \int_{y^2}^{y} e^{x/y}\,dx\,dy$$

$$= \int_0^1 \left[\frac{e^{x/y}}{1/y}\right]_{y^2}^{y} dy = \int_0^1 \left[y e^{y/y} - y e^{y^2/y}\right] dy$$

$$= \int_0^1 \left[ye - ye^y\right] dy = e\int_0^1 y\,dy - \int_0^1 y e^y\,dy$$

$$= e\left[\frac{y^2}{2}\right]_0^1 - \left[ye^y - \int e^y dy\right]_0^1 = \frac{e}{2} - \left[e - e + 1\right]$$

P (4)

P (4)
[Marks: 5]

Q:3 (a)

$$S.A = \iint_{R_{xy}} \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$$z = \sqrt{25-x^2-y^2} \Rightarrow f_x = \frac{-x}{\sqrt{25-x^2-y^2}}$$

$$f_y = \frac{-y}{\sqrt{25-x^2-y^2}}$$

$$S.A = \iint_{R_{xy}} \sqrt{\frac{25-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{25-x^2-y^2}}\, dA$$

$x^2+y^2=9$

$$= \iint_{R_x} \frac{\cancel{2}5}{\sqrt{25-y^2-x^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^{3} \frac{\cancel{2}5}{\sqrt{25-r^2}}\, r\, dr\, d\theta$$

$$= \frac{-5}{2}\int_0^{2\pi}\int_0^{3} (25-r^2)^{-1/2}(-2r)\, dr\, d\theta$$

$$= -\frac{5}{2}\int_0^{2\pi} \frac{(25-r^2)^{1/2}}{1/2}\Big]_0^3 d\theta = -5\int_0^{2\pi} \left[(16)^{\frac{1}{2}} - (25)^{\frac{1}{2}}\right] d\theta$$

$$= 5\int_0^{2\pi} d\theta = 5(2\pi) = 10\pi$$

P 5 P 5

Q. 3 (b)

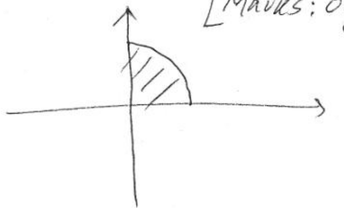
$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\,dx\,dy$$

[Marks: 05]

$0 \le \theta \le \pi/2$

$0 \le \varphi \le \pi/2$

$0 \le \rho \le 2$



$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho\left(\rho^2 \sin\varphi\right) d\rho\, d\varphi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left.\frac{\rho^4}{4}\right]_0^2 \sin\varphi\, d\varphi\, d\theta$$

$$= 4\int_0^{\pi/2} \Big[-\cos\varphi\Big]_0^{\pi/2} d\theta$$

$$= 4\int_0^{\pi/2} \left[-0+1\right] d\theta = 4\int_0^{\pi/2} d\theta$$

$$= 4\left(\pi/2\right) = 2\pi$$

#

P 6 P 6

[Marks: 4]

Q:4(a) $M = e^{3y} - y^2 \sin x \Rightarrow \frac{\partial M}{\partial y} = 3e^{3y} - 2y \sin x$

$N = 3xe^{3y} + 2y \cos x \Rightarrow \frac{\partial N}{\partial x} = 3e^{3y} - 2y \sin x$

$f_x = e^{3y} - y^2 \sin x \Rightarrow f(x,y) = xe^{3y} + y^2 \cos x + C_1$

$f_y = 3xe^{3y} + 2y \cos x \Rightarrow f(x,y) = xe^{3y} + y^2 \cos x + C_2$

$\Rightarrow f(x,y) = xe^{3y} + y^2 \cos x + C$

$$\int_{(\pi/2,0)}^{(\pi,1)} (e^{3y} - y^2 \sin x)\,dx + (3x e^{3y} + 2y \cos x)\,dy$$

$$= \Big[x e^{3y} + y^2 \cos x \Big]_{(\pi/2,0)}^{(\pi,1)}$$

$$\cancel{= \left[\pi e^0 + 0\right] - \left[\tfrac{\pi}{2} e^0 + 0\right]}$$

$$\cancel{= \pi - \tfrac{\pi}{2} = \tfrac{\pi}{2}} \quad \#$$

$= (\pi e^3 - 1) - \pi/2 = \pi(e^3 - \frac{1}{2}) - 1$

$\approx 60.529 = 60.53$

1

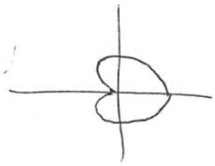
P (7) P (7)

Q. 4 (5) ~~$M = x^4 + 4$~~

[Marks: 3]

$$\oint_c \underbrace{(x^2+4)}_{M}\,dx + \underbrace{xy}_{N}\,dy = \iint_R (y-0)\,dA$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta \, r\,dr\,d\theta$$



$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1+\cos\theta} \sin\theta \;d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} (1+\cos\theta)^3(-\sin\theta)\,d\theta$$

$$= -\frac{1}{3}\;\frac{(1+\cos\theta)^4}{4}\Bigg]_0^{2\pi}$$

$$= -\frac{1}{12}\Big[(1+1) - (1+1)\Big] = 0$$

P (8) P (8)

[Marks: 5]

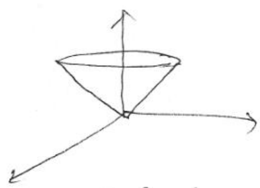
Q. 5 (a)

$$\iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dv$$

$M = x^2$, $N = y^2 - 2xy$, $P = 4z - 2yz$

$$= \iiint_Q (\cancel{2x} + \cancel{2y} - \cancel{2x} + 4 - \cancel{2y})\, dV$$

$$= 4 \iiint_Q dV$$



$$= 4 \int_0^{2\pi} \int_0^{9} \int_r^{9} r \, dz \, dr \, d\theta$$

cylindrical

$r \le z \le 9$

$x^2 + y^2 = 81$

$0 \le r \le 9$

$0 \le \theta \le 2\pi$

$$= 4 \int_0^{2\pi} \int_0^{9} r(9-r) \, dr \, d\theta$$

$$= 4 \int_0^{2\pi} \left. \frac{9r^2}{2} - \frac{r^3}{3} \right]_0^9 d\theta$$

$$= 4 \left[\frac{9(81)}{2} - \frac{(9)^3}{3} \right] 2\pi$$

$$= 8 \left[\frac{729}{2} - \frac{729}{3} \right] \pi$$

$$= \frac{8(729)}{6} \pi = 972\pi$$

#

P ⑦ P ⑨

[Marks: 08]

Q#5(l) $\oint_C \bar{F}\cdot dr = \iint_S (\text{curl}\,F)\cdot \bar{n}\, ds$

L.H.S

$\bar{F}\cdot dr = (3z+1)\,dx + (x^2-1)\,dy + (y^3+x)\,dz$

$\oint_C F\cdot dr = \oint_C (3z+1)\,dx + (x^2-1)\,dy + (y^3+x)\,dz$

$C:\quad x = 3\cos\theta,\ y = 3\sin\theta,\quad z = 0 \quad 0 \le \theta \le 2\pi$

$dx = -3\sin\theta\, d\theta,\quad dy = 3\cos\theta\, d\theta,\ dz = 0$

$= \int_0^{2\pi} (0+1)(-3\sin\theta)\,d\theta + (9\cos^2\theta - 1)(3\cos\theta\, d\theta)$

$= 0 + 27\int_0^{2\pi} \cos^3\theta\, d\theta - 3\int_0^{2\pi} \cos\theta\, d\theta$

$= 27\int_0^{2\pi} (1-\sin^2\theta)\cos\theta\, d\theta \qquad \| \; 0$

$= 27\int_0^{2\pi} \cos\theta\, d\theta - 27\int_0^{2\pi} \sin^2\theta \cos\theta\, d\theta$

$= 0 - 0 = 0$

P (10) P (10)

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z+1 & x^2-1 & y^3+x \end{vmatrix}$$

$$= i\left(3y^2 - 0\right) - j\left(1-3\right) + k\left(2x\right)$$

$$= 3y^2 i + 4 j + 2x k$$

$$\iint_S (\text{curl } F)\cdot \bar{n}\, ds = \iint_{R_{xy}} \left(-M g_x - N g_y + P\right) dA$$

$$\boxed{\begin{array}{l} M = 3y^2,\ N = 4,\ P = 2x \\ g(x,y) = 9 - x^2 - y^2 \Rightarrow g_x = -2x,\ g_y = -2y \end{array}}$$

$$= \iint_{R_{xy}} \left(+6xy^2 + 8y + 2x\right) dA$$

$$= \int_0^{2\pi}\int_0^3 \left(6 r^3 \sin^2\theta \cdot \cos\theta + 8r\sin\theta + 2r\cos\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^3 6r^4 \sin^2\theta \cdot \cos\theta\, dr + \int_0^3 8r^2 \sin\theta\, dr + \int_0^3 2r^2 \cos\theta\, d\theta\right] dr$$

$= \int_0^{2\pi} \left(\left[6 \frac{(3)^5}{} \sin^2\theta \cos \ldots \right] + \left[8 \frac{(3)^3}{} \sin \ldots \right] + \left[2 \frac{(3)^3}{} \cos \ldots \right] \right) d\ldots$