Exercise 1

Find $f(kT)$ if

$$F(z) = \frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}.$$

TABLE 13.1 Partial table of z- and s-transforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2 - 2ze^{-aT}\cos\omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos\omega T}{z^2 - 2ze^{-aT}\cos\omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

Solution 1_1

$$F(z) = \frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}.$$

Begin by dividing F(z) by z and performing a partial-fraction expansion.

$$\frac{F(z)}{z} = \frac{(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)} = \frac{A}{z-0.5} + \frac{B}{z-0.7} + \frac{C}{z-0.9}$$

Multiplying by $(z-0.5)(z-0.7)(z-0.9)$

$$(z+1)(z+2) = A(z-0.7)(z-0.9) + B\,(z-0.5)(z-0.9) + C\,(z-0.5)(z-0.7)$$

For z = 0.5 $\quad (1.5)(2.5) = A(-0.2)(z-0.4) \quad \Rightarrow A = 46.875$

For z = 0.7 $\quad (1.7)(2.7) = B(0.2)(0.2) \quad \Rightarrow B = -114.75$

For z = 0.9 $\quad (1.9)(2.9) = C(0.4)(0.2) \quad \Rightarrow C = 68.875$

Next, multiply through by z.

$$\frac{F(z)}{z} = \frac{46.875}{z-0.5} - \frac{114.75}{z-0.7} + \frac{68.875}{z-0.9} \Rightarrow F(z) = 46.875\frac{z}{z-0.5} - 114.75\frac{z}{z-0.7} + 68.875\frac{z}{z-0.9}$$

Using Table, we find the inverse z-transform of each partial fraction. the value of the time function at the sampling instants is:

$$\boxed{f^*(kT) = 46.875(0.5)^k - 114.75(0.7)^k + 68.875(0.9)^k}$$

Exercise 2

Find $G(z)$ for $G(s) = 8/(s + 4)$ in cascade with a zero order sample and hold. The sampling period is 0.25 second.

Solution 2_1

The transfer function $G(s)$ in cascade with a zero-order hold is given by: $G(s) = \frac{1-e^{-Ts}}{s} G_0(s) = \frac{1-e^{-Ts}}{s} \frac{8}{s+4}$

by moving the s in the denominator of the zero-order hold to $G_0(s)$, yielding

$z = e^{sT}$ Multiply by $\frac{z}{z}$

$$G(s) = (1-e^{-Ts}) \frac{G_0(s)}{s} \quad \text{from which} \quad G(z) = (1-z^{-1}) Z\left\{\frac{G_0(s)}{s}\right\} = \frac{z-1}{z} Z\left\{\frac{G_0(s)}{s}\right\}$$

Thus, begin the solution by finding the impulse response (inverse Laplace transform) of $\frac{G_0(s)}{s}$. Hence,

$$G_1(s) = \frac{G_0(s)}{s} = \frac{8}{s(s+4)} = \frac{A}{s} + \frac{B}{(s+4)} = \frac{2}{s} - \frac{2}{(s+4)}$$

Taking the inverse Laplace transform, we get $g_1(t) = 2\,u(t) - 2e^{-4t}$ from which $g_1(kT) = 2\,u(kT) - 2e^{-4kT}$

Using Table, we find $G_1(z) = \frac{2z}{z-1} - \frac{2z}{z-e^{-4T}}$

Substituting T = 0.25 yields $G_1(z) = Z\left\{\frac{G_0(s)}{s}\right\} = \frac{2z}{z-1} - \frac{2z}{z-0.3679} = \frac{1.264\,z}{(z-1)(z-0.3679)}$

$$\Rightarrow G(z) = \frac{z-1}{z} G_1(z) = \frac{1.264}{z-0.3679}$$

Exercise 3

Given $T(z) = N(z)/D(z)$, where $D(z) = z^3 - z^2 - 0.5z + 0.3$, use the Routh-Hurwitz criterion to find the number of z-plane poles of $T(z)$ inside, outside, and on the unit circle. Is the system stable?

Solution 3_1

$we\ have\ D(z) = z^3 - z^2 - 0.5\,z + 0.3$

Using the bilinear transformation $z = \frac{s+1}{s-1}$ and substitute into $D(z) = 0$ $we\ obtain$

$$\left(\frac{s+1}{s-1}\right)^3 - \left(\frac{s+1}{s-1}\right)^2 - 0.5\left(\frac{s+1}{s-1}\right) + 0.3 = 0$$

$$(s+1)^3 - (s-1)(s+1)^2 - 0.5(s-1)^2(s+1) + 0.3(s-1)^3 = 0$$

$$(s^3 + 3s^2 + 3s + 1) - (s^3 + s^2 - s - 1) - 0.5(s^3 - s^2 - s + 1) + 0.3(s^3 - 3s^2 + 3s - 1) = 0$$

$$-0.2\,s^3 + 1.6s^2 + 5.4s + 1.2 = 0$$

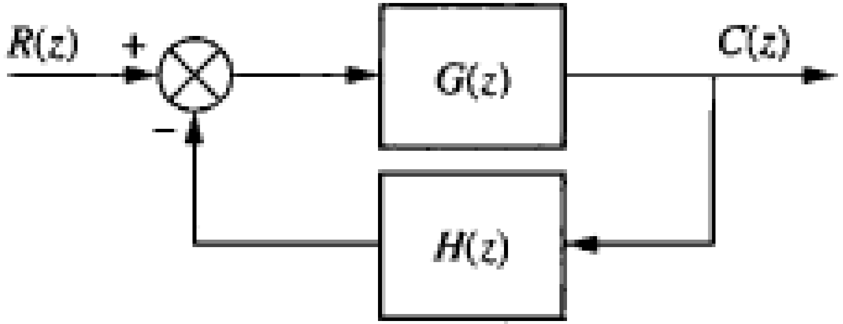
The Routh table

s^3	-0.2	5.4
s^2	1.6	1.2
s^1	5.55	0
s^0	1.2	

The Routh table shows one root in the right-half plane and two roots in the left-half-plane. Hence, T(z) has one pole outside the unit circle, no poles on the unit circle, and two poles inside the unit circle. The system is unstable because of the pole outside the unit circle.

Exercise 4

For the system of Figure where *H(z)* = 1 and



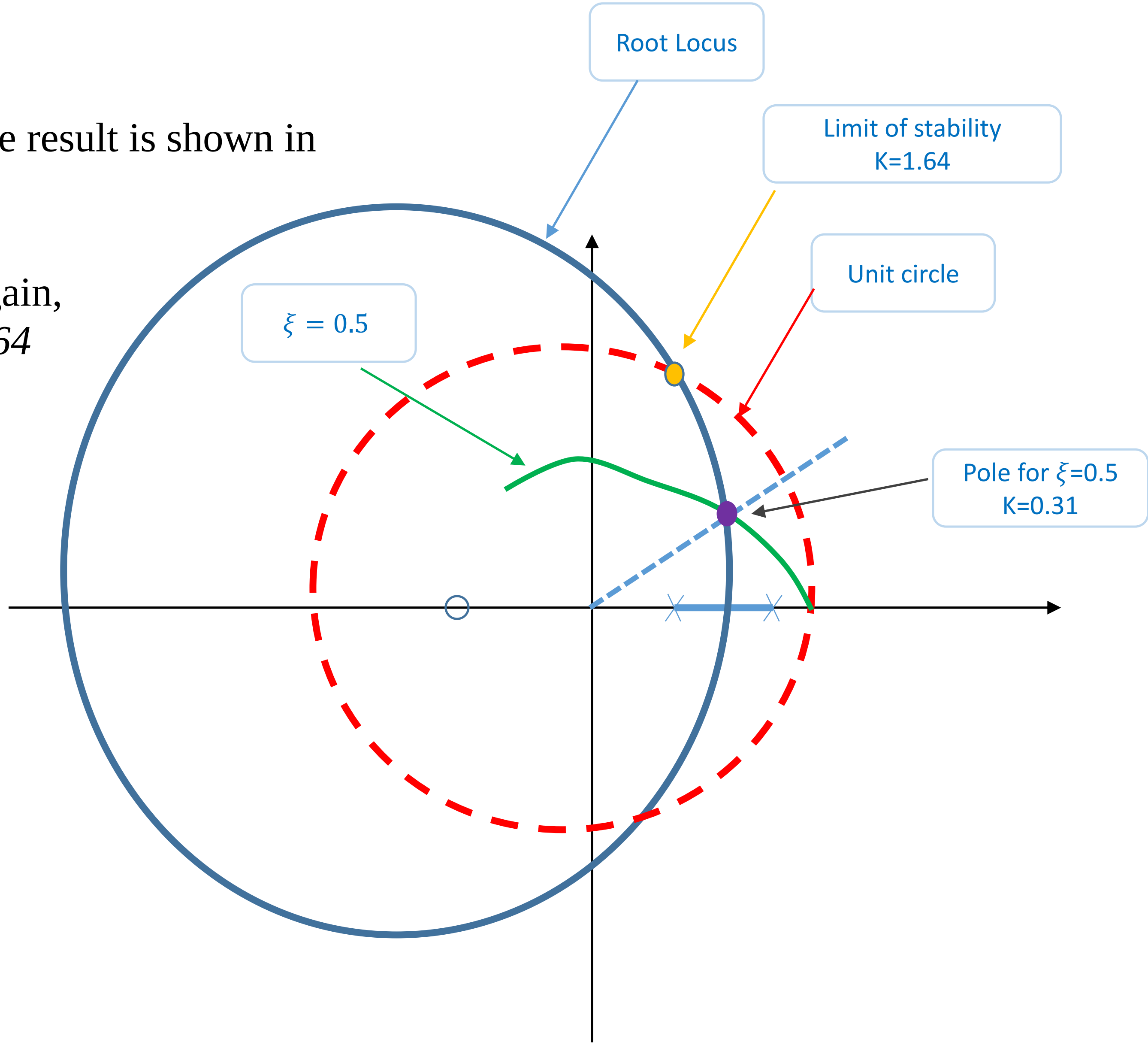
$$G(z) = \frac{K(z+0.5)}{(z-0.25)(z-0.75)}$$

1. sketch the root locus of the open-loop system.
2. determine the range of gain, K, for stability from the root locus plot.
3. find the value of gain, K, to yield a damping ratio of 0.5.

Solution 4_1

1. Treat the system as if *z* were *s*, and sketch the root locus. The result is shown in Figure

2. search along the unit circle for 180°. Identification of the gain, *K*, at this point yields the range of gain for stability. 0< *K*<*1.64*

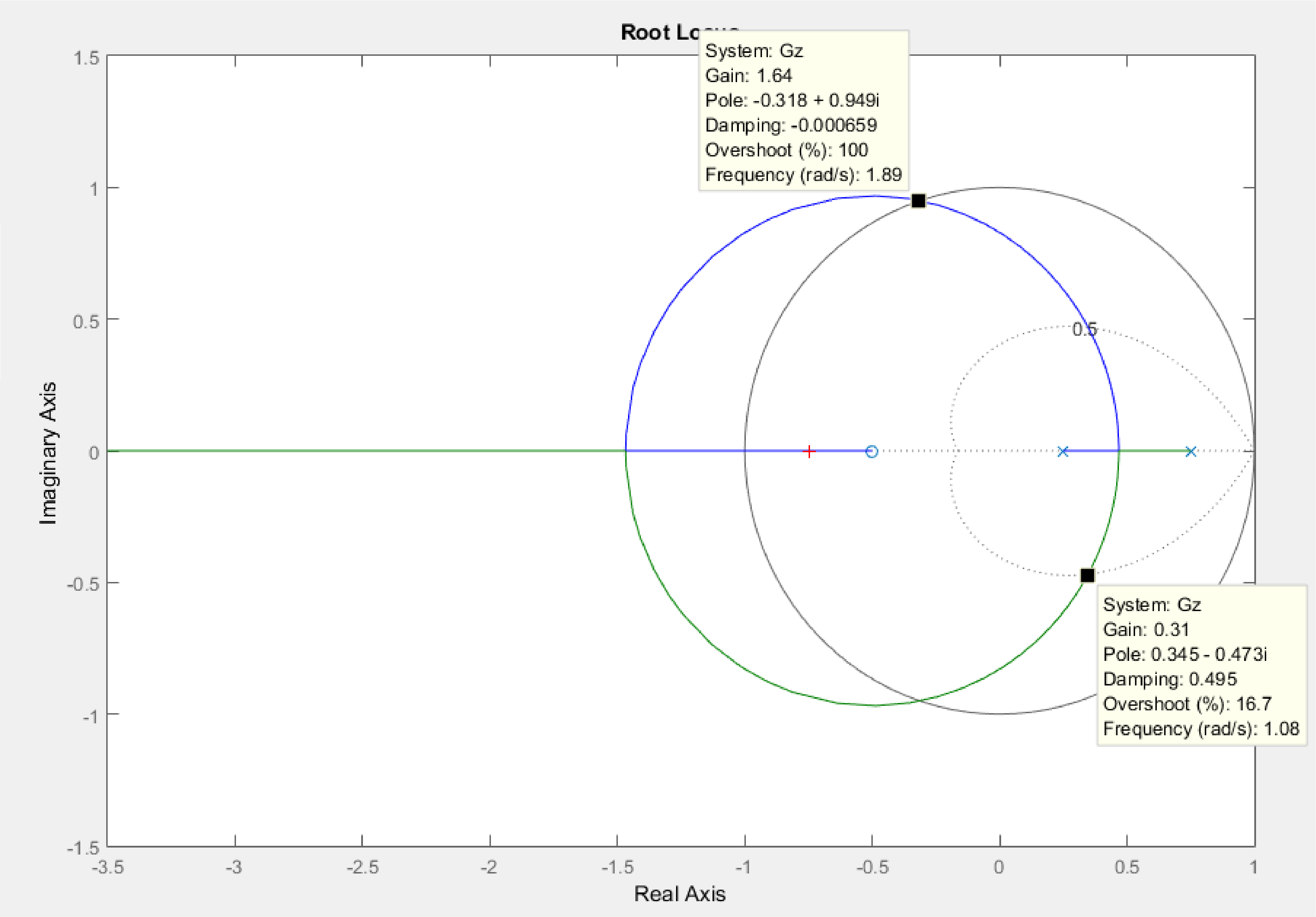


Solution 4_2

Matlab

```
clc;
Gz=zpk(-0.5,[0.25,0.75],1,[]);
rlocus(Gz)
zgrid(0.5,[])
[k,p]=rlocfind(Gz);
```

3. From the Root Locus plot the value of gain, K, to yield a damping ratio of 0.5 is K= 0.31.



Exercise 5

A lead compensator $G_c(s)$ is designed for a unity feedback system whose plant is

$$G(s) = \frac{100K}{s(s+36)(s+100)}$$

The design specifications were as follows: percent overshoot = 20%, peak time 0.1 second, and Kv = 40. In order to meet the requirements, the design yielded K = 1440 and a lead compensator,

$$G_c(s) = 2.38\,\frac{s+25.3}{s+60.2}$$

If the system is to be computer controlled, find the digital controller, $G_c(z)$ *for a sampling period of* T = 0.001 second.

Solution 5

The analog lead compensator $G_c(s) = 2.38 \dfrac{s + 25.3}{s + 60.2}$

Using the Tustin transformation. $s = \dfrac{2(z-1)}{T(z+1)}$ With T=0.001 second yields.

$$G(z) = 2.38 \frac{\left(\dfrac{2(z-1)}{T(z+1)}\right) + 25.3}{\left(\dfrac{2(z-1)}{T(z+1)}\right) + 60.2} = 2.38 \frac{2(z-1) + 25.3\,T(z+1)}{2(z-1) + 60.2\,T(z+1)} = 2.38 \frac{2(z-1) + 25.3\,(0.001)(z+1)}{2(z-1) + 60.2\,(0.001)(z+1)}$$

$$= 2.38 \frac{2.0253\,z - 1.9747}{2.0602\,z - 1.9398} \qquad \Rightarrow \boxed{G(z) = 2.34 \frac{z - 0975}{z - 0.9416}}$$

Exercise 6

Find the equivalent sampled impulse response sequence and the equivalent z-transfer function for the cascade of the two analog systems with sampled input

$$H_1(s) = \frac{1}{s+2} \qquad H_2(s) = \frac{2}{s+4}$$

1. If the systems are directly connected.

2. If the systems are separated by a sampler.

Solution

1. In the absence of samplers between the systems, the overall transfer function is

$$H(s) = \frac{2}{(s+2)(s+4)} = \frac{1}{s+2} - \frac{1}{s+4}$$

The impulse response of the cascade is $h(t) = e^{-2t} - e^{-4t}$

and the sampled impulse response is $h(kT) = e^{-2kT} - e^{-4kT}, \qquad k = 0,\ 1,\ 2, \ldots$

Thus, the z-domain transfer function is $H(z) = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-4T}} = \frac{(e^{-2T} - e^{-4T})z}{(z - e^{-2T})(z - e^{-4T})}$

2. If the analog systems are separated by a sampler, then each has a z-domain transfer function, and the transfer functions are given by

$$H_1(z) = \frac{z}{z - e^{-2T}} \qquad H_2(z) = \frac{2z}{z - e^{-4T}}$$

The overall transfer function for the cascade is $H(z) = \frac{2z^2}{(z - e^{-2T})(z - e^{-4T})}$

The partial fraction expansion of the transfer function is $H(z) = \frac{2}{e^{-2T} - e^{-4T}} \left[\frac{e^{-2T}z}{z - e^{-2T}} - \frac{e^{-4T}z}{z - e^{-4T}} \right]$

Inverse z-transforming gives the impulse response sequence

$$h(kT) = \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2T}e^{-2kT} - e^{-4T}e^{-4kT} \right] = \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2(k+1)T} - e^{-4(k+1)T} \right], \qquad k = 0,\ 1,\ 2, \ldots$$