King Saud University
College of Sciences
Mathematics Department

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Solution of the second midterm exam QMF: ACTU. 468 (25%) (two pages)

Thursday, May 4, 2017 / Sha'ban 8, 1438 (two hours 9–11 AM)

<u>**Problem 1.**</u> **(9 marks)**

1. For $0 \leq t \leq T$ we set
$$d_1(S_t, K, r, T-t, \delta) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
and
$$d_2(S_t, K, r, T-t, \delta) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
(1 mark) Find in terms of $d_1(S_t, K, r, T-t, \delta)$ and $d_2(S_t, K, r, T-t, \delta)$ the pricing Black–Scholes formula for a European call option on stock paying a dividend yield δ at any time t between 0 and maturity T.

2. **(1 mark)** Find the relation between $d_1(S_t, K, r, T-t, \delta)$ and $d_2(K, S_t, \delta, T-t, r)$ and the relation between $d_1(K, S_t, \delta, T-t, r)$ and $d_2(S_t, K, r, T-t, \delta)$.

3. **(1 mark)** Use the formula $1 - \mathbf{N}(x) = \mathbf{N}(-x)$ to find the relation between $C(S_t, K, \sigma, r, T-t, \delta)$ and $C(K, S_t, \sigma, \delta, T-t, r)$.

4. **(1 mark)** Let $S = \$100$, $K = \$90$, $\sigma = 30\%$, $r = 8\%$, $\delta = 5\%$, and $T = 1$. What is the Black–Scholes European call price?

5. **(1 mark)** Find the price of a European put where $S = \$90$, $K = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 8\%$, and $T = 1$.

6. **(1 mark)** What is the link between your answers to 4. and 5. ? Explain your answer ?

7. Consider a stock whose price is given by the Black–Scholes model, with volatility $\sigma = 30\%$ per annum and initial price $S_0 = 100$ euros. Such a stock pays a dividend of one euro in 3 months and of one euro in 9 months. The continuously compounded risk–free rate available on the market is of 4% per annum.
(1 mark) Compute the initial price of a European call option set at the money with maturity of one year.

8. **(1 mark)** Give the call–put parity corresponding to this stock.

9. **(1 mark)** Find the price of the corresponding put option?

<u>**Solution.**</u>

1. The Black–Scholes price of a European call option on a dividend–paying–stock with yield δ, at time t is give by

$$C_t = S_t e^{-\delta(T-t)}\mathbf{N}\left(d_1(S_t, K, r, T-t, \delta)\right) - Ke^{-r(T-t)}\mathbf{N}\left(d_2(K, S_t, \delta, T-t, r)\right).,$$

2. By definition of $d_2(K, S_t, \delta, T-t, r)$ we have

$$\begin{aligned} d_2(K, S_t, \delta, T-t, r) &= \frac{\ln\left(\frac{K}{S_t}\right) + \left(\delta - r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= -d_1(S_t, K, r, T-t, \delta). \end{aligned}$$

and

$$\begin{aligned} d_1(K, S_t, \delta, T-t, r) &= \frac{\ln\left(\frac{K}{S_t}\right) + \left(\delta - r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= -d_2(S_t, K, r, T-t, \delta). \end{aligned}$$

3. We know that

$$C(K, S_t, \sigma, \delta, T-t, r) = Ke^{-r(T-t)}\mathbf{N}\left(d_1(K, S_t, \delta, T-t, r)\right) - S_t e^{-\delta(T-t)}\mathbf{N}\left(d_2(K, S_t, \delta, T-t, r)\right)$$

$$\begin{aligned} &= Ke^{-r(T-t)}\mathbf{N}\left(-d_2(S_t, K, r, T-t, \delta) - S_t e^{-\delta(T-t)}\mathbf{N}\left(-d_1(S_t, K, \sigma, r, T-t, \delta)\right)\right. \\ &= Ke^{-r(T-t)}\left(1 - \mathbf{N}\left(d_2(S_t, K, r, T-t, \delta)\right) - S_t e^{-\delta(T-t)}\left(1 - \mathbf{N}\left(d_1(S_t, K, r, T-t, \delta)\right)\right)\right) \\ &= Ke^{-r(T-t)} - S_t e^{-\delta(T-t)} + S_t e^{-\delta(T-t)}\mathbf{N}\left(d_1(S_t, K, r, T-t, \delta)\right) - Ke^{-r(T-t)}\mathbf{N}\left(d_2(S_t, K, r, T-t, \delta)\right. \\ &= Ke^{-r(T-t)} - S_t e^{-\delta(T-t)} + C(S_t, K, \sigma, r, T-t, \delta) \end{aligned}$$

4. **Call option:** For $S = \$100$, $K = \$90$, $\sigma = 30\%$, $r = 8\%$, $\delta = 5\%$, and $T = 1$, we first compute d_1 and d_2.

$$d_1 = \frac{\ln\left(\frac{100}{90}\right) + \left(0.08 - 0.05 + \frac{0.3^2}{2}\right)}{0.3} = 0.6012 = 0.4345 \quad \text{and} \quad d_2 = 0.6012 - 0.3 = 0.3012.$$

Then

$$\begin{aligned} C_0 &= C(100, 90, 0.3, 0.08, 1, 0.05), \\ &= 100e^{-0.05}\mathbf{N}(0.6012) - 90e^{-0.08}\mathbf{N}(0.3012) \\ &= 100e^{-0.05} \times 0.7261 - 90e^{-0.08} \times 0.6184 = 17.692 \end{aligned}$$

5. **Put option:** For $S = \$90$, $K = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 8\%$, and $T = 1$, we first compute d_1 and d_2.

$$d_1 = \frac{\ln\left(\frac{90}{100}\right) + \left(0.05 - 0.08 + \frac{0.3^2}{2}\right)}{0.3} = -0.3012 \quad \text{and} \quad d_2 = -0.3012 - 0.3 = -0.6012$$

Then

$$\begin{aligned} P_0 &= P(90, 100, 0.3, 0.05, 1, 0.08), \\ &= 100e^{-0.05}\mathbf{N}\left(-(-0.6012)\right) - 90e^{-0.08}\mathbf{N}\left(-(-0.3012)\right) \\ &= 100e^{-0.05} \times 0.7261 - 90e^{-0.08} \times 0.6184 = 17.692. \end{aligned}$$

6. We remark

$$P(90, 100, 0.3, 0.05, 1, 0.08) = C(100, 90, 0.3, 0.08, 1, 0.05).$$

This true thanks to the duality between $d_1(S_t, K, r, T-t, \delta)$ and $d_2(K, S_t, \delta, T-t, r)$.

7. The initial price at time zero of a European call option set at the money with maturity of one year on a stock paying discrete dividend is given by that For the put option

$$\begin{aligned} C_0 &= C(F^p_{0,T}(S), F^p_{0,T}(K), \sigma, 0, T, 0) \\ &= F^p_{0,T}(S)\mathbf{N}\left(d_1\right) - F^p_{0,T}(K)\mathbf{N}\left(d_2\right), \end{aligned}$$

where

$$d_1 = \frac{\ln\left(\frac{F^p_{0,T}(S)}{F^p_{0,T}(K)}\right) + \frac{\sigma^2}{2}}{\sigma} = \frac{1}{\sigma}\ln\left(\frac{F^p_{0,T}(S)}{F^p_{0,T}(K)}\right) + \frac{\sigma}{2}$$

and

$$d_2 = \frac{\ln\left(\frac{F^p_{0,T}(S)}{F^p_{0,T}(K)}\right) - \frac{\sigma^2}{2}}{\sigma} = \frac{1}{\sigma}\ln\left(\frac{F^p_{0,T}(S)}{F^p_{0,T}(K)}\right) - \frac{\sigma}{2}$$

Now,

$$F^p_{0,T}(S) = 100 - e^{-0.04\times\frac{3}{12}} - e^{-0.04\times\frac{9}{12}} = 98.040$$

and

$$F^p_{0,T}(K) = 100e^{-0.04} = 96.079.$$

Therefore

$$d_1 = \frac{1}{0.3}\ln\left(\frac{98.040}{96.079}\right) + \frac{0.3}{2} = 0.2173$$

and

$$d_2 = \frac{1}{0.3}\ln\left(\frac{98.040}{96.079}\right) - \frac{0.3}{2} = -0.0826$$

then

$$\begin{aligned} C_0 &= 98.040\times\mathbf{N}\left(0.2173\right) - 96.079 \times \mathbf{N}\left(-0.0826\right) \\ &= 98.040\times 0.5860 - 96.079 \times 0.4670 = 12.583. \end{aligned}$$

8. The call–put parity corresponding to an asset paying discrete dividends D_1 at time t_1 and D_2 at time t_2 before maturity is given as follows

$$\begin{aligned} C_0 - P_0 &= F^p_{0,T}(S) - F^p_{0,T}(K) = S_0 - D_1e^{-rt_1} - D_2e^{-rt_2} - Ke^{-rT} \\ \Longleftrightarrow &\quad P_0 = C_0 - S_0 + D_1e^{-rt_1} + D_2e^{-rt_2} + Ke^{-rT}. \end{aligned}$$

9. The price of the corresponding put option is then

$$P_0 = 12.583 - 100 + e^{-0.04\times\frac{3}{12}} + e^{-0.04\times\frac{9}{12}} + 96.079 = 10.622.$$

Problem 2. (8 marks)

1. **(1 mark)** Recall first the Black–Scholes pricing formula of a European call option involving two currencies.
2. **(1 mark)** Recall the corresponding call–put parity.
3. One euro is currently trading for \$1.0896. The dollar–denominated continuously compounded interest rate is 2% and the euro–denominated continuously compounded interest rate is 0.5%. Volatility is 10%.
   (1 mark) Find the Black–Scholes price of a 1–year dollar–denominated euro call with strike price of \$1.1000/€.
4. **(1 mark)** Find the Black–Scholes price of a 1–year dollar–denominated euro put with strike price of \$1.1000/€.
5. **(1 mark)** What is the price of a 1–year euro–denominated dollar call with strike price of €$\frac{1}{1.1000}$/\$
6. **(1 mark)** What is the price of a 1–year euro–denominated dollar put with strike price of €$\frac{1}{1.1000}$/\$
7. **(1 mark)** What is the link between your answers to 3. and to 5. converted to dollar ?
8. **(1 mark)** What is the link between your answers to 4. and to 6. converted to euro ?

Solution:

1. The Black–Scholes pricing formula for a call on a currency is given by

$$C_0 = S_0 e^{-r_f T}\mathbf{N}(d_1) - Ke^{-rT}\mathbf{N}(d_2)$$

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{you can also use } d_2 = d_1 - \sigma\sqrt{T}$$

2. The call–put parity is $C_0 - P_0 = S_0 e^{-r_f T} - Ke^{-rT}$.

3. Our input parameters are: $S_0 = \$1.0896$, $K = \$1.1000/€$, $r = 0.02$, $r_f = 0.005$ and $\sigma = 0.1$ and $T = 1$. Therefore substituting theses parameters in the Black–Scholes formula we get

$$d_1^{\$} = \frac{\ln(\frac{1.0896}{1.1000}) + (0.02 - 0.005 + \frac{0.1^2}{2})}{0.1} = 0.105 \quad \text{and} \quad d_2^{\$} = 0.105 - 0.1 = 0.005$$

   Denote by $C_0^{\$}$ the call price of a 1–year dollar–denominated euro call with strike price of \$1.1000/€

$$\begin{aligned} C_0^{\$} &= \$S_0 e^{-r_{€}T}\mathbf{N}(d_1^{\$}) - \$Ke^{-r_{\$}T}\mathbf{N}(d_2^{\$}) \\ &= 1.0896e^{-0.005}\mathbf{N}(0.105) - 1.1e^{-0.02}\mathbf{N}(0.005) \\ &= 1.0896e^{-0.005} \times 0.5418 - 1.1e^{-0.02} \times 0.5012 = 0.04610. \end{aligned}$$

4. Denote by $P_0^{\$}$ the put price of a 1–year dollar–denominated euro put with strike price of \$1.1000/€, then Black–Scholes pricing formula for a put on a currency is given by

$$\begin{aligned} P_0^{\$} &= C_0^{\$} = \$Ke^{-r_{\$}T}\mathbf{N}(-d_1^{\$}) - \$S_0 e^{-r_{€}T}\mathbf{N}(-d_2^{\$}) \\ &= 1.1e^{-0.02}\mathbf{N}(-0.005) - 1.0896e^{-0.005}\mathbf{N}(-0.105) \\ &= 1.1e^{-0.02} \times 0.4980 - 1.0896e^{-0.005} \times 0.4582 = 0.04019 \end{aligned}$$

5. Now, with similar notation the price of a 1–year euro–denominated dollar call with strike price of $€\frac{1}{1.1000}/\$$ is given by $C_0^{€} = C(€\frac{1}{1.0896}/\$, €\frac{1}{1.1000}/\$, \sigma, r_{€}, T, r_{\$})$. Now our input parameters are: $S_0 = €\frac{1}{1.0896}/\$$, $K = €\frac{1}{1.1000}/\$$, $r_{\$} = 0.02$, $r_{€} = 0.005$ and $\sigma = 0.1$ and $T = 1$. Therefore substituting theses parameters in the Black–Scholes formula we get

$$d_1^{€} = \frac{\ln(\frac{1.1000}{1.0896}) + (0.005 - 0.02 + \frac{0.1^2}{2})}{0.1} = -0.005 \quad \text{and} \quad d_2^{€} = -0.005 - 0.1 = -0.105$$

Denote by $C_0^{€}$ the call price of a 1–year dollar–denominated euro call with strike price of $1.1000/€

$$\begin{aligned} C_0^{€} &= €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(d_1^{€}) - €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(d_2^{€}) \\ &= \frac{1}{1.0896}e^{-0.02}\mathbf{N}(-0.005) - \frac{1}{1.100}e^{-0.005}\mathbf{N}(-0.105) \\ &= \frac{1}{1.0896}e^{-0.02} \times 0.5418 - \frac{1}{1.100}e^{-0.005} \times 0.5012 = 0.0340. \end{aligned}$$

6. Denote by $P_0^{€}$ the put price of a 1–year dollar–denominated euro put with strike price of $€\frac{1}{1.1000}/\$$, then Black–Scholes pricing formula for a put on a currency is given by

$$\begin{aligned} P_0^{€} &= €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(-d_2^{€}) - €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(-d_1^{€}) \\ &= \frac{1}{1.100}e^{-0.005}\mathbf{N}(0.105) - \frac{1}{1.0896}e^{-0.02}\mathbf{N}(0.005) \\ &= \frac{1}{1.100}e^{-0.005} \times 0.5418 - \frac{1}{1.0896}e^{-0.02} \times 0.5012 = 0.0392. \end{aligned}$$

7. We have from Q3 and Q5

$$C_0^{\$} = \$S_0e^{-r_{€}T}\mathbf{N}(d_1^{\$}) - \$Ke^{-r_{\$}T}\mathbf{N}(d_2^{\$}) \quad \text{and} \quad C_0^{€} = €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(d_1^{€}) - €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(d_2^{€}).$$

Remark first that

$$d_1^{€} = \frac{\ln(\frac{K}{S_0}) + (r_{€} - r_{\$} + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -\frac{\ln(\frac{S_0}{K}) + (r_{\$} - r_{€} - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_2^{\$}$$

and also

$$d_2^{€} = \frac{\ln(\frac{K}{S_0}) + (r_{€} - r_{\$} - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -\frac{\ln(\frac{S_0}{K}) + (r_{\$} - r_{€} + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_1^{\$}.$$

Therefore

$$C_0^{€} = €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(d_1^{€}) - €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(d_2^{€}) = €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(-d_2^{\$}) - €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(-d_1^{\$}).$$

Hence

$$\begin{aligned} S_0KC_0^{€} &= Ke^{-r_{\$}T}\mathbf{N}(-d_2^{\$}) - S_0e^{-r_{€}T}\mathbf{N}(-d_1^{\$}) = Ke^{-r_{\$}T}\left(1 - \mathbf{N}(d_2^{\$})\right) - S_0e^{-r_{€}T}\left(1 - \mathbf{N}(d_1^{\$})\right) \\ &= Ke^{-r_{\$}T} - S_0e^{-r_{€}T} + S_0e^{-r_{€}T}\mathbf{N}(d_1^{\$}) - Ke^{-r_{\$}T}\mathbf{N}(d_2^{\$}) = Ke^{-r_{\$}T} - S_0e^{-r_{€}T} + C_0^{\$}. \end{aligned}$$

Conclusion

$$S_0KC_0^{€} = Ke^{-r_{\$}T} - S_0e^{-r_{€}T} + C_0^{\$}.$$

8. We have from Q4 and Q6

$$P_0^{€} = €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(-d_2^{€}) - €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(-d_1^{€}) = €\frac{1}{K}e^{-r_{€}T}\mathbf{N}(d_1^{\$}) - €\frac{1}{S_0}e^{-r_{\$}T}\mathbf{N}(d_2^{\$}).$$

Hence

$$\begin{aligned} S_0KP_0^{€} &= S_0e^{-r_{€}T}\mathbf{N}(d_1^{\$}) - Ke^{-r_{\$}T}\mathbf{N}(d_2^{\$}) = S_0e^{-r_{€}T}\mathbf{N}(-d_2^{€}) - Ke^{-r_{\$}T}\mathbf{N}(-d_1^{€}) \\ &= S_0e^{-r_{€}T}\left(1-\mathbf{N}(d_2^{€})\right) - Ke^{-r_{\$}T}\left(1-\mathbf{N}(d_1^{€})\right) \\ &= S_0e^{-r_{€}T} - Ke^{-r_{\$}T} + Ke^{-r_{\$}T}\mathbf{N}(d_1^{€}) - S_0e^{-r_{€}T}\mathbf{N}(d_2^{€}) \\ &= S_0e^{-r_{€}T} - Ke^{-r_{\$}T} + Ke^{-r_{\$}T}\mathbf{N}(-d_2^{\$}) - S_0e^{-r_{€}T}\mathbf{N}(-d_1^{\$}). \end{aligned}$$

Conclusion

$$S_0KP_0^{€} = S_0e^{-r_{€}T} - Ke^{-r_{\$}T} + P_0^{\$}.$$

$-\frac{113}{250} = 0.452 = 0.332 = 0.06$

Problem 3. (8 marks)

1. **(1 mark)** Give the formula under the objective probability (P) of a stock whose price at time T is given by the Black–Scholes model.
2. **(1 mark)** Give the formula of the continuous compounding rate of return R of a stock whose price at time T is given by the Black–Scholes model.
3. **(1 mark)** What is the distribution of R, specify the mean and the variance..
4. A stock price is currently 120. Assume that the expected return from the stock is 8% and its volatility is 20%.
   (1 mark) What is the probability distribution for the rate of return (with continuous compounding) earned over a one–year period?
5. **(1 mark)** Find the confidence interval of R at 95%.
6. **(1 mark)** Deduce the confidence interval of the stock price in one year.
7. **(1 mark)** What is the probability that a European call option on the stock with an exercise price of \$110 and a maturity date in one year will be exercised?
8. **(1 mark)** What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

Solution:

1. Under the historical probability measure or the objective probability P (real world) is given by

$$S_t = S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} \text{ for all } 0 \le t \le T$$

   where μ is the expected return on stock per year, σ is the volatility of the stock price per year and $(B_t)_{t\ge 0}$ is a standard Brownian motion under P
2. Let R denotes the continuously compounded rate of return per annum realized by the stock between times 0 and T, then $S_T = S_0e^{RT}$.

3. The rate of return R can be written as

$$R = \frac{1}{T} \ln\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma}{T} B_T,$$

hence

$$R \hookrightarrow \mathcal{N}\left(\mu - \frac{\sigma^2}{2}; \frac{\sigma^2}{T}\right)$$

This means that the continuously compounded rate of return per annum is normally distributed with mean $\mu - \frac{\sigma^2}{2}$ and standard deviation $\frac{\sigma}{\sqrt{T}}$.

4. We have $T = 1$, $\mu = 0.08$ and $\sigma = 0.2$, therefore

$$\mathcal{N}\left(\mu - \frac{\sigma^2}{2}; \sigma^2\right) = \mathcal{N}\left(0.08 - \frac{0.2^2}{2}; 0.2^2\right) = \mathcal{N}(0.06; 0.04).$$

5. the confidence interval of R at 95% is

$$]0.06 - 1.96 \times 0.2\ ;\ 0.06 + 1.96 \times 0.2[=]-0.332\ ;\ 0.452[.$$

6. Remember that $S_T = S_0 e^{RT}$, hence the confidence interval of S_T at 95% for $T = 1$ is of the form

$$\left]120e^{-0.332}\ ;\ 120e^{0.452}\right[=]86.098\ ;\ 188.57[.$$

7. The probability that a European 110–call option will be exercised in one year is

$$P(S_1 > 110) = P(\ln(S_1) > \ln(110))$$

But

$$\ln(S_1) = \ln(120) + \left(0.08 - \frac{0.2^2}{2}\right) + 0.2B_1 = 4.8475 + 0.2B_1$$

where $B_1 \hookrightarrow \mathcal{N}(0, 1)$. Therefore

$$\begin{aligned}
P(S_1 > 110) &= P(\ln(S_1) > \ln(110)) \\
&= P(4.8475 + 0.2B_1 > 4.7005) \\
&= P\left(B_1 > \frac{4.7005 - 4.8475}{0.2}\right) \\
&= P(B_1 > -0.735) \\
&= 1 - P(B_1 \leq -0.735) \\
&= 1 - F_{\mathcal{N}(0,1)}(-0.735) = F_{\mathcal{N}(0,1)}(0.735) = 0.7688.
\end{aligned}$$

8. The probability that a European 110–put option will be exercised in one year is

$$P(S_1 < 110) = 1 - P(S_1 > 110) = 0.2312.$$