Chapter 1

Discrete random variable

1.1 Discrete Probability Distributions

Definition 1 *The set of ordered pairs* $(x, f(x))$ *is a probability function, probability mass function, or probability distribution of the discrete random variable* X *if, for each possible outcome* x*,*

1. $f(x) \geq 0$,

2. $\sum_x f(x) = 1$,

3. $P(X = x) = f(x)$.

Definition 2 *The cumulative distribution function* $F(x)$ *of a discrete random variable* X *with probability distribution* $f(x)$ *is*

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), for - \infty < x < \infty$$

Definition 3 (Mean of a Random Variable) *Let* X *be a random variable with probability distribution* $f(x)$*. The mean, or expected value, of* X *is*

$$\mu = E(X) = \sum_x x f(x)$$

Example 4 *A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find* *the expected value of the number of good components in this sample.*

Example 5 *Let X represent the number of good components in the sample. The probability distribution of X is* $f(x) = \dfrac{\begin{pmatrix} 4 \\ x \end{pmatrix}\begin{pmatrix} 3 \\ 3-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$, $x = 0, 1, 2, 3$.

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = 12/7 = 1.7$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Theorem 6 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

Example 7 *Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:*

x	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Example 8 *Let X be a random variable with probability distribution as follows:*

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X - 1)^2$.

Theorem 9 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

The positive square root of the variance, σ, is called the standard deviation of X.

Example 10 *Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution*

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

1.2 Some Discrete Probability Distributions

1.2.1 Discrete Uniform Random Variable

Definition 11 (Discrete Uniform Random Variable) *A random variable X is called discrete uniform if it has a finite number of possible values, say $x_1, x_2, ..., x_n$, and $\Pr(X = x_i) = 1/n$ for all i.*

1.2.2 Binomial Distribution

Definition 12 (Bernouilli Process) *Strictly speaking, the Bernoulli process must possess the following properties:*

1. *The experiment consists of repeated trials.*
2. *Each trial results in an outcome that may be classified as a success or a failure.*
3. *The probability of success, denoted by p, remains constant from trial to trial.*
4. *The repeated trials are independent.*

Definition 13 (Binomial Distribution) *A Bernoulli trial can result in a success with probability p and a failure withprobability $q = 1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is*

$$\Pr(X = x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, ..., n.$$

Example 14 *The probability that a certain kind of component will survive a shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive.*

Solution 15 *Let X the number of components that will survive a shock test. Assuming that the tests are independent and $p = 3/4$ for each of the 4 tests, then X is a binomial distribution $Bin(4, 3/4)$. Hence,*

$$\Pr(X = 2) = \binom{4}{2}(3/4)^2(1/4)^2 \approx 0.21$$

Example 16 *The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that*

(a) at least 10 survive,

(b) from 3 to 8 survive,

and (c) exactly 5 survive?

Solution 17 *(a)* $\Pr(X \geq 10) = 1 - \Pr(X < 10)$

$= 1 - \sum_{x=0}^{9} b(x; 15, 0.4) = 1 - 0.9662 = 0.0338$

(b) $\Pr(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x; 15, 0.4)$

$= \sum_{x=0}^{8} b(x; 15, 0.4) - \sum_{x=0}^{2} b(x; 15, 0.4)$

$= 0.9050 - 0.0271 = 0.8779$

(c) $\Pr(X = 5) = b(5; 15, 0.4) = \sum_{x=0}^{5} b(x; 15, 0.4)$

$- \sum_{x=0}^{4} b(x; 15, 0.4) = 0.4032 - 0.2173 = 0.1859$

Theorem 18 *The mean and variance of the binomial distribution $B(n, p)$ are*

$$\mu = np \text{ and } \sigma^2 = npq.$$

1.3 Hypergeometric Distribution

Definition 19 (Hypergeometric Distribution) *The probability distribution of the hypergeometric random variable X, the numberof successes in a random sample of size n selected from N items of which K are labeled success and $N-K$ labeled failure, is*

$$\Pr(X=x)=\frac{\begin{pmatrix} K \\ x \end{pmatrix}\begin{pmatrix} N-K \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

Theorem 20 *The mean and variance of the hypergeometric distribution $h(N,K,n)$ are*

$$\mu=n\frac{K}{N} \text{ and } \sigma^2=n\frac{K}{N}\left(1-n\frac{K}{N}\right)\frac{N-n}{N-1}.$$

Example 21 *Lots of 40 components each are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot?*

Solution 22 *Using the hypergeometric distribution with $n=5$, $N=40$, $k=3$, and $x=1$, we find the probability of obtaining 1 defective to be*

$$h(1;40,5,3)=\frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}}=0.3011.$$

Theorem 23 (Approximation) *If n is small compared to N, then a binomial distribution $B(n,p=K/N)$ can be used to approximate the hypergeometric distribution $h(N,K,n)$.*

Example 24 *A manufacturer of automobile tires reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly blemished. If one purchases 10 of these tires at random from the distributor, what is the probability that exactly 3 are blemished?*

Solution 25 *Since $N=5000$ is large relative to the sample size $n=10$, we shall approximate the desired probability by using the binomial distribution. The probability of obtaining a* *blemished tire is* 0.2*. Therefore, the probability of obtaining exactly* 3 *blemished tires is*

$$h(3; 5000, 10, 1000) \approx b(3; 10, 0.2) = 0.8791 - 0.6778 = 0.2013.$$

1.3.1 Poisson Distribution

Definition 26 *Let X the number of outcomes occurring during a given time interval. X is called a Poisson random variable when its probability distribution is given by*

$$\Pr(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \ x = 0, 1, 2, ...,$$

where λ is the average number of outcomes.

Example 27 *During a laboratory experiment, the average number of radioactive particles passing through a counter in* 1 *millisecond is* 4*. What is the probability that* 6 *particles enter the counter in a given millisecond?*

Solution 28 *Using the Poisson distribution with $x = 6$ and $\lambda = 4$ and referring to Table A.2, we have*

$$p(6; 4) = \frac{e^{-4}4^6}{6!} = 0.1042.$$

Theorem 29 *Both the mean and the variance of the Poisson distribution $P(\lambda)$ are λ.*

Theorem 30 (Approximation) *Let X be a binomial random variable with probability distribution $B(n, p)$. When n is large ($n \to \infty$), and p small ($p \to 0$), then the poisson distribution can be used to approximate the binomial distribtion $B(n, p)$ by taking $\lambda = np$.*

Example 31 *In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is* 0.005 *and accidents are independent of each other.*
(a) What is the probability that in any given period of 400 *days there will be an accident on one day?*
(b) What is the probability that there are at most three days with an accident?

Solution 32 *Let X be a binomial random variable with $n = 400$ and $p = 0.005$. Thus, $np = 2$. Using the Poisson approximation, (a)* $\Pr(X = 1) = e^{-2}2^1 = 0.271$ *and (b)* $\Pr(X \leq 3) = e^{-2}2^x/x! = 0.857$.

Chapter 2

Continuous random variable

2.1 Probability density function

Definition 33 *The function $f(x)$ is a probability density function (pdf) for the continuous random variable X, defined over the set of real numbers, if*

1. $f(x) \geq 0$, *for all* $x \in R$.

2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1$.

3. $\Pr(a \leq X \leq b) = \int\limits_{a}^{b} f(x)dx$.

Example 34 *Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function*

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \textit{elsewhere} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $\Pr(0 \leq X \leq 1)$.

(c) Find $\Pr(0 < X < 1)$.

Definition 35 *The cumulative distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is*

$$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(t)dt, \text{ for } -\infty < x < \infty.$$

Example 36 *For the density function of Example 2, find $F(x)$, and use it to evaluate* $\Pr(0 < X \leq 1)$.

Definition 37 (Mean of a Random Variable) *Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is*

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Example 38 *For the density function of Example 2, find $E(X)$.*

Theorem 39 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Theorem 40 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)$$

Theorem 41 *Let X a random variable. The variance of a random variable X is*

$$\sigma^2 = E(X^2) - E(X)^2.$$

Theorem 42 *Let X a random variable. If a and b are constants, then $E(aX+b) = aE(X)+b$.*

Theorem 43 *The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,*

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

2.2 Some Continuous Probability Distributions

2.2.1 Continuous Uniform Distribution

Definition 44 (Uniform Distribution) *The density function of the continuous uniform random variable X on the interval $[a, b]$ is*

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \textit{elsewhere.} \end{cases}$$

Example 45 *Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.*
a) What is the probability density function?
b) What is the probability that any given conference lasts at least 3 hours?

Theorem 46 *The mean and variance of the uniform distribution are*

$$\mu = E(X) = \frac{a+b}{2} \textit{ and } \sigma^2 = \frac{(b-a)^2}{12}$$

The proofs of the theorems are left to the reader.

2.2.2 Normal Distribution

Definition 47 (Standard Normal Distribution) *The density of the standard normal distribution Z is*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}, \quad -\infty < x < \infty,$$

Theorem 48 *The mean and variance of standard normal distribution are* 0 *and* 1*, respectively. We denote the standard normal distribution by* $N(0,1)$*.*

Example 49 *Given a standard normal distribution* $N(0,1)$*, find the area under the curve that lies*
(a) to the right of $z = 1.84$
(b) between $z = -1.97$ *and* $z = 0.86$*.*

Solution 50 *(a)* 0.0329 *(b)* 0.7807*.*

Example 51 *Given a standard normal distribution* $N(0,1)$*, find the value of* k *such that*
(a) $\Pr(Z > k) = 0.3015$ *and*
(b) $P(k < Z < -0.18) = 0.4197$*.*

Solution 52 *(a)* $k = 0.52$ *(b)* $k = -2.37$*.*

Definition 53 (Normal Distribution) *The density of the normal random variable* X*, with mean* μ *and variance* σ^2*, and denoted by* $N(\mu, \sigma)$*, is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

where $\pi = 3.14159\ .\ .\ .$ *and* $e = 2.71828\ .\ .\ .\ .$

Theorem 54 *If* X *is normal random variable* $N(\mu, \sigma)$*, then the random variable* $(X - \mu)/\sigma$ *is a standard normal distribution* Z *with mean* 0 *and variance* 1*.*

Example 55 *Given a random variable* X *having a normal distribution with* $\mu = 50$ *and* $\sigma = 10$*, find the probability that* X *assumes a value between* 45 *and* 62*.*

Solution 56 *Using Table A.3, we have*
$\Pr(45 < X < 62) = \Pr(-0.5 < Z < 1.2) = \Pr(Z < 1.2) - \Pr(Z < -0.5)$
$= 0.8849 - 0.3085 = 0.5764.$

Example 57 *Given a normal distribution with* $\mu = 40$ *and* $\sigma = 6$*, find the value of* x *that has*
(a) 45% *of the area to the left*
(b) 14% *of the area to the right.*

Solution 58 *(a) We need to find a z value that leaves an area of* 0.45 *to the left. From Table A.3 we find* $\Pr(Z < -0.13) = 0.45$*, so the desired z value is* -0.13*. Hence,* $x = (6)(-0.13) + 40 = 39.22$*. (b) This time we require a z value that leaves* 0.14 *of the area to the right and hence an area of* 0.86 *to the left. Again, from Table A.3, we find* $P(Z < 1.08) = 0.86$*, so the desired z value is* 1.08 *and*

$$x = (6)(1.08) + 40 = 46.48.$$

Example 59 *An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to* 800 *hours and a standard deviation of* 40 *hours. Find the probability that a bulb burns between* 778 *and* 834 *hours.*

Solution 60 *The z values corresponding to* $x_1 = 778$ *and* $x_2 = 834$ *are*

$$z_1 = \frac{778 - 800}{40} = -0.55 \text{ and } z_2 = \frac{834 - 800}{40} = 0.85.$$

Hence,

$$\begin{aligned} \Pr(778 &< X < 834) = P(-0.55 < Z < 0.85) \\ &= P(Z < 0.85) - P(Z < -0.55) \\ &= 0.8023 - 0.2912 = 0.5111. \end{aligned}$$

Example 61 *A certain machine makes electrical resistors having a mean resistance of* 40 *ohms and a standard deviation of* 2 *ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding* 43 *ohms?*

Solution 62 *A percentage is found by multiplying the relative frequency by* 100%*. Since the relative frequency for an interval is equal to the probability of a value falling in the interval, we must find the area to the right of* $x = 43$*. This can be done by transforming* $x = 43$ *to the* *corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from* 1*. We find*

$$z = \frac{43 - 40}{2} = 1.5.$$

Therefore,$\Pr(X > 43) = \Pr(Z > 1.5) = 1 - \Pr(Z < 1.5) = 1 - 0.9332 = 0.0668$*. Hence,* 6.68% *of the resistors will have a resistance exceeding* 43 *ohms.*

2.2.3 Exponential Distribution

The exponential random variable is used when we are interested in the time of the first arrival or the time between arrival.

Definition 63 *The continuous random variable X has an exponential distribution, with parameter λ, if its density function is given by* $f(x) = \begin{cases} \lambda e^{-\lambda x}, \; x > 0 \\ 0 \qquad \text{elsewhere} \end{cases}$ *where $\lambda > 0$.*

Theorem 64 *The mean and variance of the exponential distribution are $\mu = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$.*

If X is the time of arrival of the first customer and if the average time is 30 minutes, then $\lambda = 1/30$.

Example 65 *Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure is* 5*.*
(a) If one component is installed, what is the probability that it is still functioning at the end of 8 *years?*
(b) If 5 *of these components are installed in different systems, what is the probability that at least* 2 *are still functioning at the end of* 8 *years? (Hint: use the binomial distribution).*

Solution 66 *(a)The probability that a given component is still functioning after 8 years is given by*

$$\Pr(T > 8) = \frac{1}{5}\int_{8}^{\infty} e^{-t/5}dt = e^{-8/5} \approx 0.2.$$

(b) Let X represent the number of components functioning after 8 years. X is binomial disctribution $Bin(8, 0.2)$. Then we have

$$\begin{aligned}
\Pr(X &\geq 2) = \sum_{x=2}^{5} \Pr(X = x) = 1 - \left(\sum_{x=0}^{1} \Pr(X = x) \right) \\
&= 1 - 0.7373 = 0.2627.
\end{aligned}$$

Chapter 3

Fundamental Sampling Distributions

3.1 Random sampling

Definition 67 *A population consists of the totality of the observations with which we are concerned*

Definition 68 *A sample is a subset of a population.*

In the field of statistical inference, statisticians are interested in arriving at conclusions concerning a population when it is impossible or impractical to observe the entire set of observations that make up the population. Therefore, we must depend on a subset of observations from the population to help us make **inferences** concerning that same population.

Definition 69 *A sample is a subset of a population.*

To eliminate any possibility of **bias** in the sampling procedure, it is desirable to choose a random sample in the sense that the observations are made independently and at random.

3.2 Some important statistics

Definition 70 *Any function of the random variables constituting a random sample is called a statistic.*

- Sample mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

- Sample median: $\widetilde{X} = \begin{cases} x(n+1)/2, \text{ if } n \text{ is odd,} \\ \frac{1}{2}(x_{n/2} + x_{n/2+1}), \text{ if } n \text{ is even.} \end{cases}$

- Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$

The computed value of S^2 for a given sample is denoted by s^2.

Theorem 71 *If S^2 is the variance of a random sample of size n, we may write*

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right]$$

- Sample standard deviation: $S = \sqrt{S^2}$

3.3 Sampling Distibutions

Let us consider a soft-drink machine designed to dispense, on average, 240 milliliters per drink. A company official who computes the mean of 40 drinks obtains $\overline{x} = 236$ milliliters. On the basis of this value, she decides that the machine is still dispensing drinks with an average content of $\mu = 240$ milliliters. The 40 drinks represent a sample from the infinite population of possible drinks that will be dispensed by this machine.The company official made the decision that the soft-drink machine dispenses drinks with an average content of 240 milliliters, even though the sample mean was 236 milliliters, because he knows from sampling theory that, if $\mu = 240$ milliliters, such a sample value could easily occur. In fact, if she ran similar tests, say every hour, she would expect the values of the statistic $\overline{x}$ to fluctuate above and below $\mu = 240$ milliliters. Only when the value of $\overline{x}$ is **substantially** different from 240 milliliters will the company official initiate action to adjust the machine.

Since a statistic is a random variable that depends only on the observed sample, it must have a probability distribution.

Definition 72 *The probability distribution of a statistic is called a sampling distribution.*

3.4 Sampling Distribution of Means and the Central Limit

Theorem 73 *If $X_1, X_2, ..., X_n$ are independent (?) random variables having normal distributions with means $\mu_1, \mu_2, ..., \mu_n$ and variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$, respectively, then the random variable $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has a normal distribution with mean*

$$\mu_Y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$

and variance

$$\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$$

Suppose that a random sample of n observations is taken from a normal population with mean μ and variance σ^2. Each observation X_i, $i = 1, 2, ..., n$, of the random sample will then have the same normal distribution. Hence, from Theorem 7, we conclude that

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

has a normal distribution with mean

$$\mu_{\overline{X}} = \frac{1}{n}\{\mu + \mu + ... + \mu\} = \sum_{i=1}^{n} \mu = \mu$$

and variance

$$\sigma_{\overline{X}}^2 = \frac{1}{n^2}\left\{\sigma^2 + \sigma^2 + ... + \sigma^2\right\} = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.$$

Hence, we have

Corollary 74 *If $X_1, X_2, ..., X_n$ are independent random variables having normal distributions with means μ and variances σ, then the sample mean $\overline{X}$ is normally distributed with mean equal to μ and standard deviation equal to $\sigma/\sqrt{n}$. Consequently the random variable*

$$Z = \frac{(\overline{X} - \mu)}{\sigma/\sqrt{n}}$$

is a standard normal distribution.

Theorem 75 (Central Limit Theorem) *If $\overline{X}$ is the mean of a random sample of size n taken from a population with mean μ and finite variance σ^2, then the limiting form of the distribution of*

$$Z = \frac{(\overline{X} - \mu)}{\sigma/\sqrt{n}}$$

as $n \to \infty$, is the standard normal distribution $N(0, 1)$.

The normal approximation for $\overline{X}$ will generally be good if $n \geq 30$.

Example 76 *An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours.*

Solution 77 *Here $\mu = 800$, $\sigma = 40$ and $n = 16$. The random variable $\overline{X}$ is normally distributed with mean $\mu_{\overline{X}} = \mu = 800$ and standard standard deviation $\sigma_{\overline{X}} = \sigma_X/\sqrt{n} = 10$. Then $(\overline{X} - 800)/10$ is a standard normal distribution $N(0, 1)$. Hence,*

$$\begin{aligned} \Pr(\overline{X} &< 775) = P((\overline{X} - 800)/10 < (775 - 800)/10) \\ &= P(Z < -2.5) = 0.0062. \end{aligned}$$

Example 78 *Traveling between two campuses of a university in a city via shuttle bus takes, on average, 28 minutes with a standard deviation of 5 minutes. In a given week, a bus transported passengers 40 times. What is the probability that the average transport time was more than 30 minutes?*

Solution 79 *In this case, $\mu = 28$ and $\sigma = 3$. We need to calculate the probability $Pr(\overline{X} > 30)$ with $n = 40$. Hence,*

$$\begin{aligned} Pr(\overline{X} &> 30) = \Pr\left(\frac{\overline{X} - 28}{5/\sqrt{40}} \geq \frac{30 - 28}{5/\sqrt{40}}\right) = \Pr(Z \geq 2.53) \\ &= 1 - \Pr(Z \leq 2.53) = 1 - 0.9925 = 0.0075. \end{aligned}$$

There is only a slight chance that the average time of one bus trip will exceed 30 minutes.

3.5 Sampling Distribution of the Difference between Two Means

A scientist or engineer may be interested in a comparative experiment in which two manufacturing methods, 1 and 2, are to be compared. The basis for that comparison is $\mu_1 - \mu_2$, the difference in the population means. Suppose that we have two populations, the first with mean μ_1 and variance σ_1^2, and the second with mean μ_2 and variance σ_2^2. Let the statistic $\overline{X}_1$ represent the mean of a random sample of size n_1 selected from the first population, and the statistic $\overline{X}_2$ represent the mean of a random sample of size n_2 selected from the second population, independent of the sample from the first population. What can we say about the sampling distribution of the difference $\overline{X}_1 - \overline{X}_2$ for repeated samples of size n_1 and n_2? According to Theorem 8, the variables $\overline{X}_1$ and $\overline{X}_2$ are both approximately normally distributed with means μ_1 and μ_2 and variances σ_1^2/n_1 and σ_2^2/n_2, respectively. This approximation improves as n_1 and n_2 increase. We can conclude that $\overline{X}_1 - \overline{X}_2$ is approximately normally distributed with mean

$$\mu_{\overline{X}_1-\overline{X}_2} = \mu_{\overline{X}_1} - \mu_{\overline{X}_2} = \mu_1 - \mu_2$$

and variance

$$\sigma^2_{\overline{X}_1-\overline{X}_2} = \sigma^2_{\overline{X}_1} + \sigma^2_{\overline{X}_2} = \sigma_1^2/n_1 + \sigma_2^2/n_2$$

The Central Limit Theorem can be easily extended to the two-sample, two-population case.

Theorem 80 *If independent samples of size n_1 and n_2 are drawn at random from two populations, discrete or continuous, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, then the sampling distribution of the differences of means, $\overline{X}_1 - \overline{X}_2$, is approximately normally distributed with mean and variance given by*

$$\mu_{\overline{X}_1-\overline{X}_2} = \mu_1 - \mu_2 \text{ and } \sigma^2_{\overline{X}_1-\overline{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Hence,

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

is approximately a standard normal variable.

If both n_1 and n_2 are greater than or equal to 30, the normal approximation for the distribution of $\overline{X}_1 - \overline{X}_2$ is good. Two independent experiments are run in which two different types of paint are compared.

Example 81 *Eighteen specimens are painted using type A, and the drying time, in hours, is recorded for each. The same is done with type B. The population standard deviations are both known to be* 1.0. *Assuming that the mean drying time is equal for the two types of paint, find* $P(\overline{X}_A - \overline{X}_B > 1.0)$, *where* $\overline{X}_A$ *and* $\overline{X}_B$ *are average drying times for samples of size* $n_A = n_B = 18$.

Solution 82 *From the sampling distribution of* $\overline{X}_A - \overline{X}_B$, *we know that the distribution is approximately normal with mean* $\mu_{\overline{X}_A - \overline{X}_B} = \mu_A - \mu_B = 0$ *and variance* $\sigma^2_{\overline{X}_A - \overline{X}_B} = \frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B} = 1/9$. *Corresponding to the value* $\overline{X}_A - \overline{X}_B = 1.0$, *we have*

$$z = \frac{1 - (\mu_A - \mu_B)}{\sqrt{1/9}} = \frac{1 - 0}{\sqrt{1/9}} = 3$$

so

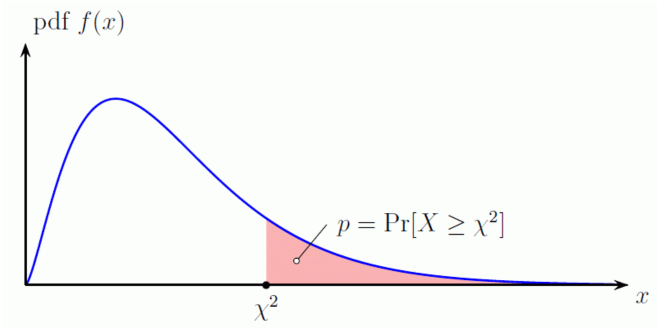
$$\Pr(Z > 3.0) = 1 - P(Z < 3.0) = 1 - 0.9987 = 0.0013.$$

Example 83 *The television picture tubes of manufacturer A have a mean lifetime of* 6.5 *years and a standard deviation of* 0.9 *year, while those of manufacturer B have a mean lifetime of* 6.0 *years and a standard deviation of* 0.8 *year. What is the probability that a random sample of* 36 *tubes from manufacturer A will have a mean lifetime that is at least* 1 *year more than the mean lifetime of a sample of* 49 *tubes from manufacturer B?*

Solution 84 *We are given the following information:*

Population 1	*Population 2*
$\mu_1 = 6.5$	$\mu_2 = 6.0$
$\sigma_1 = 0.9$	$\sigma_2 = 0.8$
$n_1 = 36$	$n_2 = 49$

If we use, the sampling distribution of $\overline{X}_1 - \overline{X}_2$ *will be approximately normal and will have a*



mean and standard deviation

$$\mu_{\overline{X}_1 - \overline{X}_2} = 6.5 - 6.0 \text{ and } \sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\tfrac{0.81}{36} + \tfrac{0.64}{49}} = 0.189$$

Hence,

$$\begin{aligned} \Pr(\overline{X}_1 - \overline{X}_2 &\geq 1.0) = P(Z > 2.65) = 1 - P(Z < 2.65) \\ &= 1 - 0.9960 = 0.0040. \end{aligned}$$

3.6 Sampling Distribution of S^2

Theorem 85 *If $X_1, X_2, ..., X_n$ an independent random sample that have the same standard normal distribution then $X = \sum_{i=1}^{n} X_i^2$ is chi-squared distribution, with $\nu = n$ degrees of freedom.*

Theorem 86 *The mean and variance of the chi-squared distribution χ^2 with ν degrees of freedom are $\mu = \nu$ and $\sigma^2 = 2\nu$.*

Theorem 87 *If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2, then the statistic*

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2}$$

has a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

Table A.5 gives values of χ^2_α for various values of α and ν. Hence, the χ^2 value with 7 degrees of freedom, leaving an area of 0.05 to the right, is $\chi^2_{0.05} = 14.067$. Owing to lack of symmetry, we must also use the tables to find $\chi^2_{0.95} = 2.167$ for $\nu = 7$.

Example 88 *For a chi-squared distribution, find*

(a)$\chi^2_{0.025}$ *when* $\nu = 15$*;*

(b)$\chi^2_{0.01}$*when* $\nu = 7$*;*

(c)$\chi^2_{0.05}$ *when* $\nu = 24$*.*

Solution 89 *(a) 27.488.(b) 18.475.(c) 36.415*

For a chi-squared distribution X, find χ^2_α such that

(a) $P(X > \chi^2_\alpha) = 0.99$ when $\nu = 4$;

(b) $P(X > \chi^2_\alpha) = 0.025$ when $\nu = 19$;

(c) $P(37.652 < X < \chi^2_\alpha) = 0.045$ when $\nu = 25$.

Solution 90 *(a)* $\chi^2_\alpha = \chi^2_{0.99} = 0.297$*.(b)* $\chi^2_\alpha = \chi^2_{0.025} = 32.852$*.(c)* $\chi^2_{0.05} = 37.652$*. Therefore,* $\alpha = 0.05 - 0.045 = 0.005$*. Hence,* $\chi^2_\alpha = \chi^2_{0.005} = 46.928$*.*

3.7 t-Distribution

Theorem 91 *Let Z be a standard normal random variable and ν a chi-squared random variable with ν degrees of freedom. If Z and ν are independent, then the distribution of the random variable T, where*

$$T = \frac{Z}{\sqrt{\nu/\nu}}$$

This is known as the t-distribution with ν degrees of freedom.

Corollary 92 *Let $X_1, X_2, ..., X_n$ be independent random variables that are all normal with mean μ and standard deviation σ. Let*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Then the random variable $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has a t-distribution with $\nu = n-1$ degrees of freedom.

Example 93 *The t-value with $\nu = 14$ degrees of freedom that leaves an area of 0.025 to the left, and therefore an area of 0.975 to the right, is*

$$t_{0.975} = -t_{0.025} = -2.145$$

Example 94 *Find $\Pr(-t_{0.025} < T < t_{0.05})$.*

Solution 95 *Since $t_{0.05}$ leaves an area of 0.05 to the right, and $-t_{0.025}$ leaves an area of 0.025 to the left, we find a total area of $1 - 0.05 - 0.025 = 0.925$ between $-t_{0.025}$ and $t_{0.05}$. Hence $\Pr(-t_{0.025} < T < t_{0.05}) = 0.925$.*

Example 96 *Find k such that $\Pr(k < T < -1.761) = 0.045$ for a random sample of size 15 selected from a normal distribution with $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$.*

Solution 97 *From Table A.4 we note that 1.761 corresponds to $t_{0.05}$ when $\nu = 14$. Therefore, $-t_{0.05} = -1.761$. Since k in the original probability statement is to the left of $-t_{0.05} = -1.761$, let $k = -t_\alpha$. Then, by using figure, we have*

$$0.045 = 0.05 - \alpha, \ \text{or } \alpha = 0.005.$$

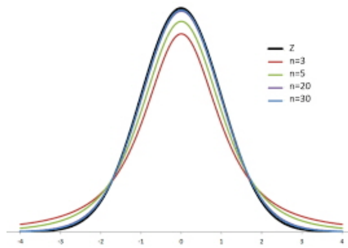


Figure 3-1: t-distribution

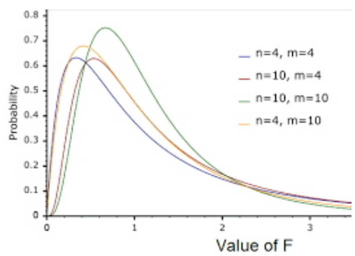


Figure 3-2: F-distribution

Hence, from Table A.4 with $\nu = 14$,

$k = -t_{0.005} = -2.977$ *and* $\Pr(-2.977 < T < -1.761) = 0.045$.

3.8 F-Distribution

The statistic F is defined to be the ratio of two independent chi-squared random variables, each divided by its number of degrees of freedom.

Theorem 98 *The random variable*

$$F = \frac{U/\nu_1}{V/\nu_2}$$

where U *and* V *are independent random variables having chi-squared distributions with* ν_1 *and* ν_2 *degrees of freedom, respectively,is the* F***-distribution*** *with* ν_1 *and* ν_2 *degrees of freedom (d.f.).*

Writing $f_\alpha(\nu_1, \nu_2)$ for f_α with ν_1 and ν_2 degrees of freedom, we obtain

$$f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_\alpha(\nu_2, \nu_1)}$$

Thus, the f-value with 6 and 10 degrees of freedom, leaving an area of 0.95 to the right, is $f_{0.95}(6, 10) = \frac{1}{f_{0.05}(10,6)} = \frac{1}{4.06} = 0.246$.

3.8.1 The F-Distribution with Two Sample Variances

Suppose that random samples of size n_1 and n_2 are selected from two normal populations with variances σ_1^2 and σ_2^2, respectively. From Theorem 16, we know that

$$\chi_1^2 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2} \text{ and } \chi_2^2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2}$$

are random variables having chi-squared distributions with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. Furthermore, since the samples are selected at random, we are dealing with independent random variables. Then, using Theorem 24 with $\chi_1^2 = U$ and $\chi_2^2 = V$, we obtain the following result.

Theorem 99 *If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 taken from normal populations with variances σ_1^2 and σ_2^2, respectively, then*

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

3.8.2 Example

For an F-distribution, find

(a) $f_{0.05}$ with $\nu_1 = 7$ and $\nu_2 = 15$;

(b) $f_{0.05}$ with $\nu_1 = 15$ and $\nu_2 = 7$:

(c) $f_{0.01}$ with $\nu_1 = 24$ and $\nu_2 = 19$;

(d) $f_{0.95}$ with $\nu_1 = 19$ and $\nu_2 = 24$;

(e) $f_{0.99}$ with $\nu_1 = 28$ and $\nu_2 = 12$.

Solution 100 *(a) 2.71.(b) 3.51.(c) 2.92.(d) 1/2.11 = 0.47.(e) 1/2.90 = 0.34.*

3.9 Sampling Distribution of Proportions and the Central Limit

In many situations the use of the sample proportion is easier and more reliable because, unlike the mean, the proportion does not depend on the population variance, which is usually an unknown quantity. We will represent the sample proportion by $\widehat{P}$ and the population proportion by p. Construction of the sampling distribution of the sample proportion is done in a manner similar to that of the mean. One has $\widehat{P} = X/n$ where X is a number of success for a sample of size n. It is clear that X is a binomial distribution $Bin(n,p)$. Its mean $\mu_X = np$ and its variance $\sigma_X^2 = np(1-p)$. Consequently:

Theorem 101 *The mean $\mu_{\widehat{p}}$ of the sample distribution $\widehat{P}$ is equal to the true population proportion p, and its variance $\sigma_{\widehat{p}}^2$ is equal to $p(1-p)/n$.*

Theorem 102 (Theorem Cemtral Limit) *If $np \geq 5$ and $n(1-p) \geq 5$, then the random variable $\widehat{P}$ is approximation a normal distribution with mean $\mu_{\widehat{p}} = p$ and standard deviation (or standard error) $\sigma_{\widehat{p}} = \sqrt{p(1-p)/n}$. Hence*

$$Z = \frac{\widehat{P} - p}{\sqrt{p(1-p)/n}}$$

is approxiamately a standard normal distribution.

Example 103 *In the mid seventies, according to a report by the National Center for Health Statistics,* 19.4 *percent of the adult U.S. male population was obese. What is the probability that in a simple random sample of size 150 from this population fewer than 15 percent will be obese?*

Solution 104 *Here $n = 150$, $p = 0.194$. Since $np \geq 5$ and $n(1-p) \geq 5$, hence*

$$Z = \frac{\widehat{P} - 0.194}{\sqrt{0.194(1-0.194)/150}} = \frac{\widehat{P} - 0.194}{0.032}$$

is approxiamately a standard normal distribution.
$\Pr(\widehat{P} \leq 0.15) = \Pr(\frac{\widehat{P}-0.194}{0.03} \leq \frac{0.15-0.194}{0.03}) \simeq \Pr(Z \leq -1.37) = 0.0853.$

3.10 Sampling Distribution of the Difference between Two Proportions

In some applications there are two actual physical dichotomous populations so that p_1 denotes the population success proportion for population one and p_2 denotes the population success proportion for population two. The sampling distribution of the difference between two sample proportions is constructed in a manner similar to the difference between two means. Independent random samples of size n_1 and n_2 are drawn from two populations of dichotomous variables where the proportions of observations with the character of interest in the two populations are p_1 and p_2, respectively.

Theorem 105 *The mean $\mu_{\widehat{p}_1-\widehat{p}_2}$ of the sample distribution of the difference between two sample proportions $\widehat{P}_1 - \widehat{P}_2$ is equal to the difference $p_1 - p_2$ between the true population proportions, and its variance $\sigma^2_{\widehat{p}_1-\widehat{p}_2}$ will be equal to $p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2$.*

Theorem 106 *If $n_1p_1 \geq 5$, $n_1(1-p_1) \geq 5$, $n_2p_2 \geq 5$, $n_2(1-p_2)$, then the random variable $\widehat{P}_1 - \widehat{P}_2$ is approximation a normal distribution with mean $\mu_{\widehat{p}_1-\widehat{p}_2} = p_1 - p_2$ and standard deviation (or standard error) $\sigma_{\widehat{p}} = \sqrt{p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2}$. Hence*

$$Z = \frac{\widehat{P}_1 - \widehat{P}_2 - p_1 - p_2}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$

is approxiamately a standard normal distribution.

Example 107 *Suppose that there are two large high schools, each with more than 2000 students, in a certain town. At School 1, 70% of students did their homework last night. Only 50% of the students at School 2 did their homework last night. The counselor at School 1 takes a sample random sample of 100 students and records the proportion that did homework. School 2's counselor takes a sample random sample of 200 students and records the proportion that did homework. Find the probability of getting a difference in sample proportion $\widehat{P}_1 - \widehat{P}_2$ of 0.10 or less from the two surveys.*

Solution 108 *Here $p_1 = 0.7$, $p_2 = 0.5$, $n_1 = 100$ and $n_2 = 200$. It is clear that $n_1p_1 \geq 5$, $n_1(1-p_1) \geq 5$, $n_2p_2 \geq 5$, $n_2(1-p_2)$. Also $\mu_{\widehat{p}_1-\widehat{p}_2} = p_1 - p_2 = 0.2$ and $\sigma_{\widehat{p}} = \sqrt{p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2} =$*

0.058. *Hence,*

$$Z = \frac{\widehat{P}_1 - \widehat{P}_2 - p_1 - p_2}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} = \frac{\widehat{P}_1 - \widehat{P}_2 - 0.2}{0.058}$$

is approximately a standard normal.

$\Pr(\widehat{P}_1 - \widehat{P}_2 \leq 0.10) = \Pr(\frac{\widehat{P}_1 - \widehat{P}_2 - 0.2}{0.68} \leq \frac{0.10-0.2}{0.68}) \simeq \Pr(Z \leq -1.72) = 0.0427.$

Chapter 4

One and Two-Sample Estimation Problems

4.1 One- and Two-Sample Estimation Problems

4.1.1 Introduction

In previous chapters, we emphasized sampling properties of the sample mean and variance. The purpose of these presentations is to build a foundation that allows us to draw conclusions about the population parameters from experimental data.

4.1.2 Classical Methods of Estimation

A point estimate of some population parameter θ is a single value $\widehat{\theta}$ of a statistic $\widehat{\Theta}$. For example, the value $\overline{x}$ of the statistic $\overline{X}$, computed from a sample of size n, is a point estimate of the population parameter μ. Similarly, $\widehat{p} = x/n$ is a point estimate of the true proportion p for a binomial experiment.

An estimator is not expected to estimate the population parameter without error. We do not expect $\overline{X}$ to estimate μ exactly, but we certainly hope that it is not far off.

Unbiased Estimator

What are the desirable properties of a "good" decision function that would influence us to choose one estimator rather than another? Let $\widehat{\Theta}$ be an estimator whose value $\widehat{\theta}$ is a point estimate of some unknown population parameter θ. Certainly, we would like the sampling distribution of $\widehat{\Theta}$ to have a mean equal to the parameter estimated. An estimator possessing this property is said to be unbiased.

Definition 109 *A statistic $\widehat{\Theta}$ is said to be an unbiased estimator of the parameter θ if $\mu_{\widehat{\Theta}}$ $= E(\widehat{\Theta}) = \theta$.*

Example 110 *Show that S^2 is an unbiased estimator of the parameter σ^2. Hint: $(X_i - \overline{X}) = (X_i - \mu) - (\overline{X} - \mu)$.*

Variance of a Point Estimator

If $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ are two unbiased estimators of the same population parameter θ, we want to choose the estimator whose sampling distribution has the smaller variance.

Hence, if $\sigma^2_{\widehat{\theta}_1} < \sigma^2_{\widehat{\theta}_2}$, we say that $\widehat{\Theta}_1$ is a more efficient estimator of θ than $\widehat{\Theta}_1$.

Definition 111 *If we consider all possible unbiased estimators of some parameter θ, the one with the smallest variance is called the most efficient estimator of θ.*

Interval Estimation

Even the most efficient unbiased estimator is unlikely to estimate the population parameter exactly. There is no reason we should expect a **point estimate** from a given sample to be exactly equal to the population parameter it is supposed to estimate. There are many situations in which it is preferable to determine an interval within which we would expect to find the value of the parameter. Such an interval is called an interval estimate. An interval estimate of a population parameter θ is an interval of the form $\widehat{\theta}_L < \theta < \widehat{\theta}_U$, where $\widehat{\theta}_l$ and $\widehat{\theta}_U$ depend on the value of the statistic $\widehat{\Theta}$ for a particular sample and also on the sampling distribution of $\widehat{\Theta}$.

4.2 Single Sample: Estimating the Mean

The sampling distribution of $\overline{X}$ is centered at μ, and in most applications the variance is smaller than that of any other estimators of μ. Thus, the sample mean $\overline{x}$ will be used as a point estimate for the population mean μ.

Let us now consider the interval estimate of μ. If our sample is selected from a normal population or, failing this, if n is sufficiently large, we can establish a confidence interval for μ by considering the sampling distribution of $\overline{X}$.

Definition 112 (Interval on μ, σ^2) *If $\overline{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by*

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example 113 *The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.*

Solution 114 *The point estimate of μ is $\overline{x} = 2.6$. The z-value leaving an area of 0.025 to the right, and therefore an area of 0.975 to the left, is $z_{0.025} = 1.96$ (Table A.3). Hence, the 95% confidence interval is*

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right)$$

which reduces to $2.50 < \mu < 2.70$. To find a 99% confidence interval, we find the z-value leaving an area of 0.005 to the right and 0.995 to the left. From Table A.3 again, $z_{0.005} = 2.575$, and the 99% confidence interval is

$$2.6 - (2.575)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (2.575)\left(\frac{0.3}{\sqrt{36}}\right)$$

or simply

$$2.47 < \mu < 2.73.$$

The error in estimating μ by $\overline{x}$ is the absolute value of the difference between μ and $\overline{x}$, and we can be $100(1-\alpha)\%$ confident that this difference will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Theorem 115 *If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.*

Theorem 116 *If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed a specified amount e when the sample size is*

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$$

Example 117 *How large a sample is required if we want to be 95% confident that our estimate of μ in Example 5 is off by less than 0.05?*

Solution 118 *The population standard deviation is $\sigma = 0.3$. Then,*

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2 = 138.3.$$

Therefore, we can be 95% confident that a random sample of size 139 will provide an estimate $\overline{x}$ differing from μ by an amount less than 0.05.

The reader should recall learning in Chapter 3 that if we have a random sample from a normal distribution, then the random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a Student t-distribution with $n-1$ degrees of freedom. Here S is the sample standard deviation. In this situation, with σ unknown, T can be used to construct a confidence interval on μ.

Definition 119 (Interval on μ, σ^2) *If $\overline{x}$ and s are the mean and standard deviation of a random sample of size n from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is*

$$\overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $\nu = n-1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Example 120 *The contents of seven similar containers of sulfuric acid are* $9.8, 10.2, 10.4, 9.8, 10.0, 10.2, 9.6$ *liters. Find a* 95% *confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.*

Solution 121 *The sample mean and standard deviation for the given data are*

$$\overline{x} = 10.0 \text{ and } s = 0.283.$$

Using Table A.4, we find $t_{0.025} = 2.447$ for $v = 6$ degrees of freedom. Hence, the 95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right)$$

which reduces to $9.74 < \mu < 10.26$.

Concept of a Large-Sample Confidence Interval

Often statisticians recommend that even when normality cannot be assumed, σ is unknown, and $n \geq 30$, s can replace σ and the confidence interval

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

may be used. This is often referred to as a large-sample confidence interval.

Example 122 *Scholastic Aptitude Test (SAT) mathematics scores of a random sample of* 500 *high school seniors in the state of Texas are collected, and the sample mean and standard* *deviation are found to be* 501 *and* 112*, respectively. Find a* 99% *confidence interval on the mean SAT mathematics score for seniors in the state of Texas.*

4.3 Standard Error of a Point Estimate

We indicated earlier that a measure of the quality of an unbiased estimator is its variance. The variance of $\overline{X}$ is

$$\sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$$

Thus, the standard deviation of $\overline{X}$, or standard error of $\overline{X}$, is $\sigma/\sqrt{n}$. Simply put, the standard error of an estimator is its standard deviation. For $\overline{X}$, the computed confidence limit

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ is written } \overline{x} \pm z_{\alpha/2} \text{ s.e.}(\overline{x})$$

In the case where σ is unknown and sampling is from a normal distribution, s replaces σ and the estimated standard error $s/\sqrt{n}$ is involved. Thus, the confidence limits on μ are limit

$$\overline{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} \text{ is written } \overline{x} \pm t_{\alpha/2} \text{ s.e.}(\overline{x})$$

4.4 Two Samples: Estimating the Difference between Two Means

Theorem 123 ***Confidence Interval for $\mu_1 - \mu_2$, σ_1^2 and σ_2^2 known***
If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2 from populations with known variances σ_1^2 and σ_2^2, respectively, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\overline{x}_1 - \overline{x}_2) - z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example 124 *A study was conducted in which two types of engines, A and B, were compared. Gas mileage, in miles per gallon, was measured. Fifty experiments were conducted using engine type A and* 75 *experiments were done with engine type B. The gasoline used and other conditions were held constant. The average gas mileage was* 36 *miles per gallon for engine A and* 42 *miles* *per gallon for engine B. Find a 96% confidence interval on $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileages for engines A and B, respectively. Assume that the population standard deviations are 6 and 8 for engines and B, respectively.*

Solution 125 *The point estimate of $\mu_B - \mu_A$ is $\overline{x}_B - \overline{x}_A = 42 - 36 = 6$. Using $\alpha = 0.04$, we find $z_{0.02} = 2.05$ from Table A.3. Hence, with substitution in the formula above, the 96% confidence interval is*

$$6 - 2.05\sqrt{\frac{64}{75} + \frac{36}{50}} < \mu_B - \mu_A < 6 + 2.05\sqrt{\frac{64}{75} + \frac{36}{50}}$$

or simply $3.43 < \mu_B - \mu_A < 8.57$.

Variances Unknown but Equal

Consider the case where σ_1^2 and σ_2^2 are unknown and $\sigma_1^2 = \sigma_1^2$ $(= \sigma^2)$. A point estimate of the unknown common variance σ^2 can be obtained by pooling the sample variances. Denoting the pooled estimator by S_p^2, we have the following.

Definition 126 (of Variance) $S_p^2 = \frac{(n_1-1)S_1^2+(n_1-1)S_2^2}{(n_1+n_2-1)}$

Theorem 127 ***Confidence Interval for $\mu_1 - \mu_2$, $\sigma_1^2 = \sigma_2^2$ but Both Uknown***
If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2, respectively, from approximately normal populations with unknown but equal variances, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\overline{x}_1 - \overline{x}_2) - t_{\frac{\alpha}{2}} s_p\sqrt{\tfrac{1}{n_1} + \tfrac{1}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\frac{\alpha}{2}} s_p\sqrt{\tfrac{1}{n_1} + \tfrac{1}{n_2}},$$

where where s_p is the pooled estimate of the population standard deviation and $t_{\alpha/2}$ is the t-value with $\nu = n_1 + n_2 - 2$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Example 128 *Two independent sampling stations, statoin 1 and station 2, were chosen for a study on pollution. For 12 monthly samples collected at station 1, the species diversity index had a mean value $\overline{x}_1 = 3.11$ and a standard deviation $s_1 = 0.771$, while 10 monthly samples collected at the station 2 had a mean index value $\overline{x}_2 = 2.04$ and a standard deviation $s_2 = 0.448$. Find a*

90% confidence interval for the difference between the population means for the two locations, assuming that the populations are approximately normally distributed with equal variances.

Solution 129 *Let μ_1 and μ_2 represent the population means, respectively, for the species diversity indices at the downstream and upstream stations. We wish to find a 90% confidence interval for $\mu_1 - \mu_2$. Our point estimate of $\mu_1 - \mu_2$ is*

$$\overline{x}_1 - \overline{x}_2 = 3.11 - 2.04 = 1.07.$$

The pooled estimate, s_p^2, of the common variance, σ^2, is

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_1-1)s_2^2}{(n_1+n_2-1)} = \frac{(11)(0.7712)+(9)(0.4482)}{12+10-2} = 0.417.$$

Taking the square root, we obtain $s_p = 0.646$. Using $\alpha = 0.1$, we find in Table A.4 that $t_{0.05} = 1.725$ for $\nu = n_1 + n_2 - 2 = 20$ degrees of freedom. Therefore, the 90% confidence interval for $\mu_1 - \mu_2$ is

$$\begin{aligned} 1.07+1.725(0.646)\sqrt{\tfrac{1}{12}+\tfrac{1}{10}} &< \mu_1-\mu_2 \\ &< 1.07+1.725(0.646)\sqrt{\tfrac{1}{12}+\tfrac{1}{10}} \end{aligned}$$

which simplifies to $0.593 < \mu_1 - \mu_2 < 1.547$.

4.5 Paired Observations

Now we shall consider estimation procedures for the difference of two means when the samples are not independent and the variances of the two populations are not necessarily equal. The situation considered here deals with a very special experimental condition, namely that of paired observations. For example, if we run a test on a new diet using 15 individuals, the weights before and after going on the diet form the information for our two samples. The two populations are "before" and "after," and the experimental unit is the individual. Obviously, the observations in a pair have something in common. To determine if the diet is effective, we consider the differences $d_1, d_2, ..., d_n$ in the paired observations. These differences are the values of a random sample $D_1, D_2, ..., D_n$ from a population of differences that we shall assume to be normally distributed with mean $\mu_D = \mu_1 - \mu_2$ and variance σ_D^2. We estimate σ_D^2 by σ_d^2, the variance of the differences that constitute our sample. The point estimator of μ_D is given by $\overline{D}$.

Theorem 130 ***Confidence Interval for*** $\mu_D = \mu_1 - \mu_2$***, for Paired Observations***
If $\overline{d}$ *and* s_d *are the mean and standard deviation, respectively, of the normally distributed differences of* n *random pairs of measurements, a* $100(1-\alpha)\%$ *confidence interval for* $\mu_D = \mu_1 - \mu_2$ *is*

$$\overline{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu < \overline{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}},$$

where $t_{\alpha/2}$ *is the t-value with* $\nu = n-1$ *degrees of freedom, leaving an area of* $\alpha/2$ *to the right.*

Example 131 *A study published in Chemosphere reported the levels of the dioxin TCDD of* 10 *Massachusetts Vietnam veterans who were possibly exposed to Agent Orange. The TCDD levels in plasma and in fat tissue are listed in Table 1. Find a* 95% *confidence interval for* $\mu_1 - \mu_2$*, where* μ_1 *and* μ_2 *represent the true mean TCDD levels in plasma and in fat tissue, respectively. Assume the distribution of the differences to be approximately normal.*

Veteran	TCDD levels in Plasma	TCDD levels in Fat Tissue	d_i
1	2.5	4.9	-2.4
2	3.1	5.9	-2.8
3	2.1	4.4	-2.3
4	3.5	6.9	-3.4
5	3.1	7.0	-3.9
6	1.8	4.2	-2.4
7	6.0	10.0	-4.0
8	3.0	5.5	-2.5
9	36.0	41.0	-5.0
10	4.7	4.4	0.3

Solution 132 *The point estimate of* μ_D *is* $\overline{d} = -2.84$*. The standard deviation,* s_d*, of the sample differences is* 1.42*. Using* $\alpha = 0.05$*, we find in Table A.4 that* $t_{0.025} = 2.262$ *for* $\nu =$

$n-1=9$ degrees of freedom. Therefore, the 95% confidence interval is

$$-2.84-(2.262)\left(\frac{1.42}{\sqrt{10}}\right)<\mu_D<-2.84+(2.262)\left(\frac{1.42}{\sqrt{10}}\right)$$

or simply $-3.85<\mu_D<-1.82$.

4.6 Single Sample: Estimating a Proportion

A point estimator of the proportion p in a binomial experiment is given by the statistic $\widehat{P}=X/n$, where X represents the number of successes in n trials. Therefore,

Definition 133 *the sample proportion $\widehat{p}=x/n$ will be used as the point estimate of the*

parameter p.

Theorem 134 (Large-Sample Confidence Intervals for p) *If $\widehat{p}$ is the proportion of successes in a random sample of size n and $\widehat{q}=1-\widehat{p}$, an approximate $100(1-\alpha)\%$ confidence interval, for the binomial parameter p is given by*

$$\widehat{p}-z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}<p<\widehat{p}+z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example 135 *In a random sample of $n=500$ families owning television sets in the city of Hamilton, Canada, it is found that $x=340$ subscribe to HBO. Find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.*

Solution 136 *The point estimate of p is $\widehat{p}=340/500=0.68$. Using Table A.3, we find that $z_{0.025}=1.96$. Therefore, the 95% confidence interval for p is*

$$0.68-1.96\sqrt{\frac{(0.68)(0.32)}{500}}<p<0.68+1.96\sqrt{\frac{(0.68)(0.32)}{500}}$$

which simplifies to $0.6391<p<0.7209$.

Theorem 137 *If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}$.*

Choice of Sample Size

Let us now determine how large a sample is necessary to ensure that the error in estimating p will be less than a specified amount e. By Theorem 23, we must choose n such that $z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}} = e$.

Theorem 138 *If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will be less than a specified amount e when the sample size is approximately*

$$n = \frac{z_{\alpha/2}^2\widehat{pq}}{e^2}$$

Example 139 *How large a sample is required if we want to be 95% confident that our estimate of p in Example 21 is within* 0.02 *of the true value?*

Solution 140 *Let us treat the* 500 *families as a preliminary sample, providing an estimate $\widehat{p}$ $= 0.68$. Then,*

$$n = \frac{(1.96)^2(0.68)(0.32)}{0.02^2} = 2089.8 \approx 2090$$

Occasionally, it will be impractical to obtain an estimate of p to be used for determining the sample size for a specified degree of confidence. If this happens, we use the following theorem.

Theorem 141 *If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will not exceed than a specified amount e when the sample size is approximately*

$$n = \frac{z_{\alpha/2}^2}{4e^2}$$

Example 142 *How large a sample is required if we want to be at least* 95% *confident that our estimate of p in Example 21 is within* 0.02 *of the true value?*

Solution 143 *Let assume that no preliminary sample has been taken to provide an estimate of p. Consequently, we can be at least* 95% *confident that our sample proportion will not differ from the true proportion by more than* 0.02 *if we choose a sample of size*

$$n = \frac{(1.96)^2}{4(0.02)^2} = 2401$$

Comparing the results of Examples 28 and 29, we see that information concerning p, provided by a preliminary sample or from experience, enables us to choose a smaller sample while maintaining our required degree of accuracy.

4.7 Two Samples: Estimating the Difference between Two Proportions

Consider the problem where we wish to estimate the difference between two binomial parameters p_1 and p_2. For example, p_1 might be the proportion of smokers with lung cancer and p_2 the proportion of nonsmokers with lung cancer, and the problem is to estimate the difference between these two proportions.

Theorem 144 ***Large-Sample Confidence Interval for*** $p_1 - p_2$
If $\widehat{p}_1$ *and* $\widehat{p}_2$ *are the proportions of successes in random samples of sizes* n_1 *and* n_2*, respectively,* $\widehat{q}_1 = 1 - \widehat{p}_1$*, and* $\widehat{q}_2 = 1 - \widehat{p}_2$*, an approximate* $100(1-\alpha)\%$ *confidence interval for the difference of two binomial parameters,* $p_1 - p_2$*, is given by*

$$(\widehat{p}_1 - \widehat{p}_2)\text{-}z_{\frac{\alpha}{2}}\sqrt{\tfrac{\widehat{p}_1\widehat{q}_1}{n_1} + \tfrac{\widehat{p}_2\widehat{q}_2}{n_1}} < p_1 - p_2 < (\widehat{p}_1 - \widehat{p}_2) + z_{\frac{\alpha}{2}}\sqrt{\tfrac{\widehat{p}_1\widehat{q}_1}{n_1} + \tfrac{\widehat{p}_2\widehat{q}_2}{n_1}}$$

Example 145 *A certain change in a process for manufacturing component parts is being considered. Samples are taken under both the existing and the new process so as to determine if the new process results in an improvement. If* 75 *of* 1500 *items from the existing process are found to be defective and* 80 *of* 2000 *items from the new process are found to be defective, find a* 90% *confidence interval for the true difference in the proportion of defectives between the existing and the new process.*

Solution 146 *Let* p_1 *and* p_2 *be the true proportions of defectives for the existing and new processes, respectively. Hence,* $\widehat{p}_1 = 75/1500 = 0.05$ *and* $\widehat{p}_2 = 80/2000 = 0.04$*, and the point estimate of* $p_1 - p_2$ *is*

$$\widehat{p}_1 - \widehat{p}_2 = 0.05 - 0.04 = 0.01$$

Using Table A.3, we find $z_{0.05} = 1.645$. *Therefore, substituting into the formula, with*

$$1.645\sqrt{\frac{(0.05)(0.95)}{1500} + \frac{(0.04)(0.96)}{2000}} = 0.0117,$$

we find the 90% *confidence interval to be* $-0.0017 < p_1 - p_2 < 0.0217$.

4.8 Single Sample: Estimating the Variance

If a sample of size n is drawn from a normal population with variance σ^2 and the sample variance s^2 is computed, we obtain a value of the statistic S^2. This computed sample variance is used as a point estimate of σ^2. Hence, the statistic S^2 is called an estimator of σ^2. An interval estimate of σ^2 can be established by using the statistic

$$X = \frac{(n-1)S^2}{\sigma^2}$$

the statistic X has a chi-squared distribution with $n-1$ degrees of freedom when samples are chosen from a normal population.

Theorem 147 (Confidence Interval for σ^2) *If s^2 is the variance of a random sample of size n from a normal population, a* $100(1-\alpha)\%$ *confidence interval for σ^2 is*

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are χ^2-values with $\nu = n-1$ degrees of freedom, leaving areas of $\alpha/2$ and $1-\ \alpha/2$, respectively, to the right.

An approximate $100(1-\alpha)\%$ confidence interval for σ is obtained by taking the square root of each endpoint of the interval for σ^2.

Example 148 *The following are the weights, in decagrams, of 10 packages of grass seed distributed by a certain company:* $46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0$. *Find a* 95% *confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.*

Solution 149 *First we find* $s^2 = 0.286$. *To obtain a* 95% *confidence interval, we choose* $\alpha = 0.05$. *Then, using Table A.5 with* $\nu = 9$ *degrees of freedom, we find* $\chi^2_{.025} = 19.023$ *and* $\chi^2_{.975} = 2.700$. *Therefore, the* 95% *confidence interval for* σ^2 *is*

$$\frac{(9)(0.286)}{19.023} < \sigma^2 < \frac{(9)(0.286)}{2.700}$$

or simply $0.135 < \sigma^2 < 0.953$.

4.9 Two Samples: Estimating the Ratio of Two Variances

A point estimate of the ratio of two population variances σ_1^2/σ_2^2 is given by the ratios s_1^2/s_2^2 of the sample variances. Hence, the statistic S_1^2/S_2^2 is called an estimator of σ_1^2/σ_2^2 . If σ_1^2 andσ_2^2 are the variances of normal populations, we can establish an interval estimate of σ_1^2/σ_2^2 by using the statistic

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

According to Theorem 25 of chapter 3, the random variable F has an F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

Theorem 150 (Confidence Interval for σ_1^2/σ_2^2) *If* s_1^2 *and* s_2^2 *are the variances of independent samples of sizes* n_1 *and* n_2*, respectively,from normal populations, then a* $100(1-\alpha)\%$ *confidence interval for* σ_1^2/σ_2^2 *is*

$$\frac{s_1^2}{s_2^2}\frac{1}{f_{\alpha/2}(\nu_1,\nu_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{\alpha/2}(\nu_2,\nu_1)$$

where $f_{\alpha/2}(\nu_1,\nu_2)$ *is an* f*-value with* $\nu_1 = n_1 - 1$ *and* $\nu_2 = n_2 - 1$ *degrees of freedom, leaving an area of* $\alpha/2$ *to the right, and* $f_{\alpha/2}(\nu_2,\nu_1)$ *is a similar* f*-value with* $\nu_2 = n_2 - 1$ *and* $\nu_1 = n_1 - 1$ *degrees of freedom.*

an approximate $100(1-\alpha)\%$ confidence interval for σ_1/σ_2 is obtained by taking the square root of each endpoint of the interval for σ_1^2/σ_2^2.

Example 151 *A study was conducted to estimate the difference in the amounts of the chemical orthophosphorus measured at two different stations. Fifteen samples were collected from station*

1, and 12 *samples were obtained from station 2. The* 15 *samples from station 1 had an average orthophosphorus content of* 3.84 *milligrams per liter and a standard deviation of* 3.07 *milligrams per liter, while the* 12 *samples from station 2 had an average content of* 1.49 *milligrams per liter and a standard deviation of* 0.80 *milligram per liter. Determine a 98% confidence interval for* σ_1^2/σ_2^2 *and for* σ_1/σ_2*, where* σ_1^2 *and* σ_2^2 *are the variances of the populations of orthophosphorus contents at station* 1 *and station* 2*, respectively.*

Solution 152 *We have* $n = 15$*,* $n_2 = 12$*,* $s_1 = 3.07$*, and* $s_2 = 0.80$*. For a* 98% *confidence interval,* $\alpha = 0.02$*. Interpolating in Table A.6, we find* $f_{0.01}(14, 11) \approx 4.30$ *and* $f_{0.01}(11, 14) \approx 3.87$*. Therefore, the* 98% *confidence interval for* σ_1^2/σ_2^2 *is*

$$\left(\frac{3.07^2}{0.80^2}\right)\left(\frac{1}{4.30}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \left(\frac{3.07^2}{0.80^2}\right)(3.87),$$

which simplifies to $3.425 < \frac{\sigma_1^2}{\sigma_2^2} < 56.991$*. Taking square roots of the confidence limits, we find that a* 98% *confidence interval for* σ_1/σ_2 *is*

$$1.851 < \frac{\sigma_1}{\sigma_2} < 7.549.$$

Since this interval does not allow for the possibility of σ_1/σ_2 *being equal to* 1*, we were correct in assuming that* $\sigma_1 \neq \sigma_2$ *(and* $\sigma_1^2 \neq \sigma_2^2$*).*

Table A.1 Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	***r***	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
1	**0**	0.9000	0.8000	0.7500	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	**0**	0.8100	0.6400	0.5625	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100
	1	0.9900	0.9600	0.9375	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	**0**	0.7290	0.5120	0.4219	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010
	1	0.9720	0.8960	0.8438	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280
	2	0.9990	0.9920	0.9844	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	**0**	0.6561	0.4096	0.3164	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001
	1	0.9477	0.8192	0.7383	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037
	2	0.9963	0.9728	0.9492	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523
	3	0.9999	0.9984	0.9961	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	**0**	0.5905	0.3277	0.2373	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000
	1	0.9185	0.7373	0.6328	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005
	2	0.9914	0.9421	0.8965	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086
	3	0.9995	0.9933	0.9844	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815
	4	1.0000	0.9997	0.9990	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	**0**	0.5314	0.2621	0.1780	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000
	1	0.8857	0.6554	0.5339	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001
	2	0.9842	0.9011	0.8306	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013
	3	0.9987	0.9830	0.9624	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159
	4	0.9999	0.9984	0.9954	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143
	5	1.0000	0.9999	0.9998	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	**0**	0.4783	0.2097	0.1335	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	
	1	0.8503	0.5767	0.4449	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000
	2	0.9743	0.8520	0.7564	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002
	3	0.9973	0.9667	0.9294	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027
	4	0.9998	0.9953	0.9871	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257
	5	1.0000	0.9996	0.9987	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497
	6		1.0000	0.9999	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217
	7			1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

							p				
n	r	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
8	**0**	0.4305	0.1678	0.1001	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	
	1	0.8131	0.5033	0.3671	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	
	2	0.9619	0.7969	0.6785	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000
	3	0.9950	0.9437	0.8862	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004
	4	0.9996	0.9896	0.9727	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050
	5	1.0000	0.9988	0.9958	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381
	6		0.9999	0.9996	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869
	7		1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695
	8				1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	**0**	0.3874	0.1342	0.0751	0.0404	0.0101	0.0020	0.0003	0.0000		
	1	0.7748	0.4362	0.3003	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	
	2	0.9470	0.7382	0.6007	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000
	3	0.9917	0.9144	0.8343	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001
	4	0.9991	0.9804	0.9511	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009
	5	0.9999	0.9969	0.9900	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083
	6	1.0000	0.9997	0.9987	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530
	7		1.0000	0.9999	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252
	8			1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126
	9					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	**0**	0.3487	0.1074	0.0563	0.0282	0.0060	0.0010	0.0001	0.0000		
	1	0.7361	0.3758	0.2440	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	
	2	0.9298	0.6778	0.5256	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	
	3	0.9872	0.8791	0.7759	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000
	4	0.9984	0.9672	0.9219	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001
	5	0.9999	0.9936	0.9803	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016
	6	1.0000	0.9991	0.9965	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128
	7		0.9999	0.9996	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702
	8		1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639
	9				1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513
	10					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	**0**	0.3138	0.0859	0.0422	0.0198	0.0036	0.0005	0.0000			
	1	0.6974	0.3221	0.1971	0.1130	0.0302	0.0059	0.0007	0.0000		
	2	0.9104	0.6174	0.4552	0.3127	0.1189	0.0327	0.0059	0.0006	0.0000	
	3	0.9815	0.8389	0.7133	0.5696	0.2963	0.1133	0.0293	0.0043	0.0002	
	4	0.9972	0.9496	0.8854	0.7897	0.5328	0.2744	0.0994	0.0216	0.0020	0.0000
	5	0.9997	0.9883	0.9657	0.9218	0.7535	0.5000	0.2465	0.0782	0.0117	0.0003
	6	1.0000	0.9980	0.9924	0.9784	0.9006	0.7256	0.4672	0.2103	0.0504	0.0028
	7		0.9998	0.9988	0.9957	0.9707	0.8867	0.7037	0.4304	0.1611	0.0185
	8		1.0000	0.9999	0.9994	0.9941	0.9673	0.8811	0.6873	0.3826	0.0896
	9			1.0000	1.0000	0.9993	0.9941	0.9698	0.8870	0.6779	0.3026
	10					1.0000	0.9995	0.9964	0.9802	0.9141	0.6862
	11						1.0000	1.0000	1.0000	1.0000	1.0000

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	r	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
12	**0**	0.2824	0.0687	0.0317	0.0138	0.0022	0.0002	0.0000			
	1	0.6590	0.2749	0.1584	0.0850	0.0196	0.0032	0.0003	0.0000		
	2	0.8891	0.5583	0.3907	0.2528	0.0834	0.0193	0.0028	0.0002	0.0000	
	3	0.9744	0.7946	0.6488	0.4925	0.2253	0.0730	0.0153	0.0017	0.0001	
	4	0.9957	0.9274	0.8424	0.7237	0.4382	0.1938	0.0573	0.0095	0.0006	0.0000
	5	0.9995	0.9806	0.9456	0.8822	0.6652	0.3872	0.1582	0.0386	0.0039	0.0001
	6	0.9999	0.9961	0.9857	0.9614	0.8418	0.6128	0.3348	0.1178	0.0194	0.0005
	7	1.0000	0.9994	0.9972	0.9905	0.9427	0.8062	0.5618	0.2763	0.0726	0.0043
	8		0.9999	0.9996	0.9983	0.9847	0.9270	0.7747	0.5075	0.2054	0.0256
	9		1.0000	1.0000	0.9998	0.9972	0.9807	0.9166	0.7472	0.4417	0.1109
	10				1.0000	0.9997	0.9968	0.9804	0.9150	0.7251	0.3410
	11					1.0000	0.9998	0.9978	0.9862	0.9313	0.7176
	12						1.0000	1.0000	1.0000	1.0000	1.0000
13	**0**	0.2542	0.0550	0.0238	0.0097	0.0013	0.0001	0.0000			
	1	0.6213	0.2336	0.1267	0.0637	0.0126	0.0017	0.0001	0.0000		
	2	0.8661	0.5017	0.3326	0.2025	0.0579	0.0112	0.0013	0.0001		
	3	0.9658	0.7473	0.5843	0.4206	0.1686	0.0461	0.0078	0.0007	0.0000	
	4	0.9935	0.9009	0.7940	0.6543	0.3530	0.1334	0.0321	0.0040	0.0002	
	5	0.9991	0.9700	0.9198	0.8346	0.5744	0.2905	0.0977	0.0182	0.0012	0.0000
	6	0.9999	0.9930	0.9757	0.9376	0.7712	0.5000	0.2288	0.0624	0.0070	0.0001
	7	1.0000	0.9988	0.9944	0.9818	0.9023	0.7095	0.4256	0.1654	0.0300	0.0009
	8		0.9998	0.9990	0.9960	0.9679	0.8666	0.6470	0.3457	0.0991	0.0065
	9		1.0000	0.9999	0.9993	0.9922	0.9539	0.8314	0.5794	0.2527	0.0342
	10			1.0000	0.9999	0.9987	0.9888	0.9421	0.7975	0.4983	0.1339
	11				1.0000	0.9999	0.9983	0.9874	0.9363	0.7664	0.3787
	12					1.0000	0.9999	0.9987	0.9903	0.9450	0.7458
	13						1.0000	1.0000	1.0000	1.0000	1.0000
14	**0**	0.2288	0.0440	0.0178	0.0068	0.0008	0.0001	0.0000			
	1	0.5846	0.1979	0.1010	0.0475	0.0081	0.0009	0.0001			
	2	0.8416	0.4481	0.2811	0.1608	0.0398	0.0065	0.0006	0.0000		
	3	0.9559	0.6982	0.5213	0.3552	0.1243	0.0287	0.0039	0.0002		
	4	0.9908	0.8702	0.7415	0.5842	0.2793	0.0898	0.0175	0.0017	0.0000	
	5	0.9985	0.9561	0.8883	0.7805	0.4859	0.2120	0.0583	0.0083	0.0004	
	6	0.9998	0.9884	0.9617	0.9067	0.6925	0.3953	0.1501	0.0315	0.0024	0.0000
	7	1.0000	0.9976	0.9897	0.9685	0.8499	0.6047	0.3075	0.0933	0.0116	0.0002
	8		0.9996	0.9978	0.9917	0.9417	0.7880	0.5141	0.2195	0.0439	0.0015
	9		1.0000	0.9997	0.9983	0.9825	0.9102	0.7207	0.4158	0.1298	0.0092
	10			1.0000	0.9998	0.9961	0.9713	0.8757	0.6448	0.3018	0.0441
	11				1.0000	0.9994	0.9935	0.9602	0.8392	0.5519	0.1584
	12					0.9999	0.9991	0.9919	0.9525	0.8021	0.4154
	13					1.0000	0.9999	0.9992	0.9932	0.9560	0.7712
	14						1.0000	1.0000	1.0000	1.0000	1.0000

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	***r***	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
15	**0**	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000				
	1	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000			
	2	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000		
	3	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001		
	4	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	
	5	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	
	6	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	
	7	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000
	8		0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003
	9		0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022
	10		1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127
	11			1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556
	12				1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841
	13					1.0000	0.9995	0.9948	0.9647	0.8329	0.4510
	14						1.0000	0.9995	0.9953	0.9648	0.7941
	15							1.0000	1.0000	1.0000	1.0000
16	**0**	0.1853	0.0281	0.0100	0.0033	0.0003	0.0000				
	1	0.5147	0.1407	0.0635	0.0261	0.0033	0.0003	0.0000			
	2	0.7892	0.3518	0.1971	0.0994	0.0183	0.0021	0.0001			
	3	0.9316	0.5981	0.4050	0.2459	0.0651	0.0106	0.0009	0.0000		
	4	0.9830	0.7982	0.6302	0.4499	0.1666	0.0384	0.0049	0.0003		
	5	0.9967	0.9183	0.8103	0.6598	0.3288	0.1051	0.0191	0.0016	0.0000	
	6	0.9995	0.9733	0.9204	0.8247	0.5272	0.2272	0.0583	0.0071	0.0002	
	7	0.9999	0.9930	0.9729	0.9256	0.7161	0.4018	0.1423	0.0257	0.0015	0.0000
	8	1.0000	0.9985	0.9925	0.9743	0.8577	0.5982	0.2839	0.0744	0.0070	0.0001
	9		0.9998	0.9984	0.9929	0.9417	0.7728	0.4728	0.1753	0.0267	0.0005
	10		1.0000	0.9997	0.9984	0.9809	0.8949	0.6712	0.3402	0.0817	0.0033
	11			1.0000	0.9997	0.9951	0.9616	0.8334	0.5501	0.2018	0.0170
	12				1.0000	0.9991	0.9894	0.9349	0.7541	0.4019	0.0684
	13					0.9999	0.9979	0.9817	0.9006	0.6482	0.2108
	14					1.0000	0.9997	0.9967	0.9739	0.8593	0.4853
	15						1.0000	0.9997	0.9967	0.9719	0.8147
	16							1.0000	1.0000	1.0000	1.0000

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	***r***	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
17	**0**	0.1668	0.0225	0.0075	0.0023	0.0002	0.0000				
	1	0.4818	0.1182	0.0501	0.0193	0.0021	0.0001	0.0000			
	2	0.7618	0.3096	0.1637	0.0774	0.0123	0.0012	0.0001			
	3	0.9174	0.5489	0.3530	0.2019	0.0464	0.0064	0.0005	0.0000		
	4	0.9779	0.7582	0.5739	0.3887	0.1260	0.0245	0.0025	0.0001		
	5	0.9953	0.8943	0.7653	0.5968	0.2639	0.0717	0.0106	0.0007	0.0000	
	6	0.9992	0.9623	0.8929	0.7752	0.4478	0.1662	0.0348	0.0032	0.0001	
	7	0.9999	0.9891	0.9598	0.8954	0.6405	0.3145	0.0919	0.0127	0.0005	
	8	1.0000	0.9974	0.9876	0.9597	0.8011	0.5000	0.1989	0.0403	0.0026	0.0000
	9		0.9995	0.9969	0.9873	0.9081	0.6855	0.3595	0.1046	0.0109	0.0001
	10		0.9999	0.9994	0.9968	0.9652	0.8338	0.5522	0.2248	0.0377	0.0008
	11		1.0000	0.9999	0.9993	0.9894	0.9283	0.7361	0.4032	0.1057	0.0047
	12			1.0000	0.9999	0.9975	0.9755	0.8740	0.6113	0.2418	0.0221
	13				1.0000	0.9995	0.9936	0.9536	0.7981	0.4511	0.0826
	14					0.9999	0.9988	0.9877	0.9226	0.6904	0.2382
	15					1.0000	0.9999	0.9979	0.9807	0.8818	0.5182
	16						1.0000	0.9998	0.9977	0.9775	0.8332
	17							1.0000	1.0000	1.0000	1.0000
18	**0**	0.1501	0.0180	0.0056	0.0016	0.0001	0.0000				
	1	0.4503	0.0991	0.0395	0.0142	0.0013	0.0001				
	2	0.7338	0.2713	0.1353	0.0600	0.0082	0.0007	0.0000			
	3	0.9018	0.5010	0.3057	0.1646	0.0328	0.0038	0.0002			
	4	0.9718	0.7164	0.5187	0.3327	0.0942	0.0154	0.0013	0.0000		
	5	0.9936	0.8671	0.7175	0.5344	0.2088	0.0481	0.0058	0.0003		
	6	0.9988	0.9487	0.8610	0.7217	0.3743	0.1189	0.0203	0.0014	0.0000	
	7	0.9998	0.9837	0.9431	0.8593	0.5634	0.2403	0.0576	0.0061	0.0002	
	8	1.0000	0.9957	0.9807	0.9404	0.7368	0.4073	0.1347	0.0210	0.0009	
	9		0.9991	0.9946	0.9790	0.8653	0.5927	0.2632	0.0596	0.0043	0.0000
	10		0.9998	0.9988	0.9939	0.9424	0.7597	0.4366	0.1407	0.0163	0.0002
	11		1.0000	0.9998	0.9986	0.9797	0.8811	0.6257	0.2783	0.0513	0.0012
	12			1.0000	0.9997	0.9942	0.9519	0.7912	0.4656	0.1329	0.0064
	13				1.0000	0.9987	0.9846	0.9058	0.6673	0.2836	0.0282
	14					0.9998	0.9962	0.9672	0.8354	0.4990	0.0982
	15					1.0000	0.9993	0.9918	0.9400	0.7287	0.2662
	16						0.9999	0.9987	0.9858	0.9009	0.5497
	17						1.0000	0.9999	0.9984	0.9820	0.8499
	18							1.0000	1.0000	1.0000	1.0000

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	***r***	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
19	**0**	0.1351	0.0144	0.0042	0.0011	0.0001					
	1	0.4203	0.0829	0.0310	0.0104	0.0008	0.0000				
	2	0.7054	0.2369	0.1113	0.0462	0.0055	0.0004	0.0000			
	3	0.8850	0.4551	0.2631	0.1332	0.0230	0.0022	0.0001			
	4	0.9648	0.6733	0.4654	0.2822	0.0696	0.0096	0.0006	0.0000		
	5	0.9914	0.8369	0.6678	0.4739	0.1629	0.0318	0.0031	0.0001		
	6	0.9983	0.9324	0.8251	0.6655	0.3081	0.0835	0.0116	0.0006		
	7	0.9997	0.9767	0.9225	0.8180	0.4878	0.1796	0.0352	0.0028	0.0000	
	8	1.0000	0.9933	0.9713	0.9161	0.6675	0.3238	0.0885	0.0105	0.0003	
	9		0.9984	0.9911	0.9674	0.8139	0.5000	0.1861	0.0326	0.0016	
	10		0.9997	0.9977	0.9895	0.9115	0.6762	0.3325	0.0839	0.0067	0.0000
	11		1.0000	0.9995	0.9972	0.9648	0.8204	0.5122	0.1820	0.0233	0.0003
	12			0.9999	0.9994	0.9884	0.9165	0.6919	0.3345	0.0676	0.0017
	13			1.0000	0.9999	0.9969	0.9682	0.8371	0.5261	0.1631	0.0086
	14				1.0000	0.9994	0.9904	0.9304	0.7178	0.3267	0.0352
	15					0.9999	0.9978	0.9770	0.8668	0.5449	0.1150
	16					1.0000	0.9996	0.9945	0.9538	0.7631	0.2946
	17						1.0000	0.9992	0.9896	0.9171	0.5797
	18							0.9999	0.9989	0.9856	0.8649
	19							1.0000	1.0000	1.0000	1.0000
20	**0**	0.1216	0.0115	0.0032	0.0008	0.0000					
	1	0.3917	0.0692	0.0243	0.0076	0.0005	0.0000				
	2	0.6769	0.2061	0.0913	0.0355	0.0036	0.0002				
	3	0.8670	0.4114	0.2252	0.1071	0.0160	0.0013	0.0000			
	4	0.9568	0.6296	0.4148	0.2375	0.0510	0.0059	0.0003			
	5	0.9887	0.8042	0.6172	0.4164	0.1256	0.0207	0.0016	0.0000		
	6	0.9976	0.9133	0.7858	0.6080	0.2500	0.0577	0.0065	0.0003		
	7	0.9996	0.9679	0.8982	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	
	8	0.9999	0.9900	0.9591	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	
	9	1.0000	0.9974	0.9861	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	
	10		0.9994	0.9961	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000
	11		0.9999	0.9991	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001
	12		1.0000	0.9998	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004
	13			1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024
	14				1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113
	15					0.9997	0.9941	0.9490	0.7625	0.3704	0.0432
	16					1.0000	0.9987	0.9840	0.8929	0.5886	0.1330
	17						0.9998	0.9964	0.9645	0.7939	0.3231
	18						1.0000	0.9995	0.9924	0.9308	0.6083
	19							1.0000	0.9992	0.9885	0.8784
	20								1.0000	1.0000	1.0000

Table A.2 Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865
4		1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977
5				1.0000	1.0000	1.0000	0.9999	0.9998	0.9997
6							1.0000	1.0000	1.0000

	μ								
r	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
0	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8		1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9			1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10				0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11				1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12					1.0000	0.9999	0.9997	0.9992	0.9980
13						1.0000	0.9999	0.9997	0.9993
14							1.0000	0.9999	0.9998
15								1.0000	0.9999
16									1.0000

Table A.2 (continued) Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**5.5**	**6.0**	**6.5**	**7.0**	**7.5**	**8.0**	**8.5**	**9.0**	**9.5**
0	0.0041	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001
1	0.0266	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008
2	0.0884	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042
3	0.2017	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149
4	0.3575	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403
5	0.5289	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885
6	0.6860	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649
7	0.8095	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687
8	0.8944	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918
9	0.9462	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218
10	0.9747	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453
11	0.9890	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520
12	0.9955	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364
13	0.9983	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981
14	0.9994	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400
15	0.9998	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665
16	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823
17	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911
18		1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957
19			1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980
20						0.9999	0.9998	0.9996	0.9991
21						1.0000	0.9999	0.9998	0.9996
22							1.0000	0.9999	0.9999
23								1.0000	0.9999
24									1.0000

Table A.2 (continued) Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**10.0**	**11.0**	**12.0**	**13.0**	**14.0**	**15.0**	**16.0**	**17.0**	**18.0**
0	0.0000	0.0000	0.0000						
1	0.0005	0.0002	0.0001	0.0000	0.0000				
2	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000		
3	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000
4	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002	0.0001
5	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007	0.0003
6	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021	0.0010
7	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054	0.0029
8	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126	0.0071
9	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261	0.0154
10	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491	0.0304
11	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847	0.0549
12	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350	0.0917
13	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009	0.1426
14	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808	0.2081
15	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.2867
16	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677	0.3751
17	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640	0.4686
18	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550	0.5622
19	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363	0.6509
20	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055	0.7307
21	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615	0.7991
22	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047	0.8551
23	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367	0.8989
24	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594	0.9317
25		0.9999	0.9997	0.9990	0.9974	0.9938	0.9869	0.9748	0.9554
26		1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848	0.9718
27			0.9999	0.9998	0.9994	0.9983	0.9959	0.9912	0.9827
28			1.0000	0.9999	0.9997	0.9991	0.9978	0.9950	0.9897
29				1.0000	0.9999	0.9996	0.9989	0.9973	0.9941
30					0.9999	0.9998	0.9994	0.9986	0.9967
31					1.0000	0.9999	0.9997	0.9993	0.9982
32						1.0000	0.9999	0.9996	0.9990
33							0.9999	0.9998	0.9995
34							1.0000	0.9999	0.9998
35								1.0000	0.9999
36									0.9999
37									1.0000

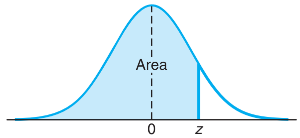


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

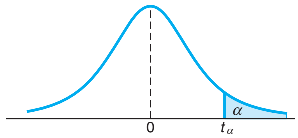


Table A.4 Critical Values of the t-Distribution

	α						
v	**0.40**	**0.30**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

Table A.4 (continued) Critical Values of the t-Distribution

	α						
v	**0.02**	**0.015**	**0.01**	**0.0075**	**0.005**	**0.0025**	**0.0005**
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

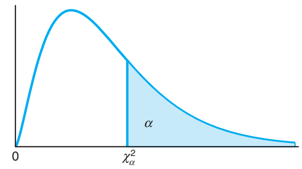


Table A.5 Critical Values of the Chi-Squared Distribution

	α									
v	**0.995**	**0.99**	**0.98**	**0.975**	**0.95**	**0.90**	**0.80**	**0.75**	**0.70**	**0.50**
1	0.0^4393	0.0^3157	0.0^3628	0.0^3982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.689	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.878	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336
40	20.707	22.164	23.838	24.433	26.509	29.051	32.345	33.66	34.872	39.335
50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	42.942	44.313	49.335
60	35.534	37.485	39.699	40.482	43.188	46.459	50.641	52.294	53.809	59.335

Table A.5 (continued) Critical Values of the Chi-Squared Distribution

	α									
v	**0.30**	**0.25**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.001**
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
60	65.226	66.981	68.972	74.397	79.082	83.298	84.58	88.379	91.952	99.608

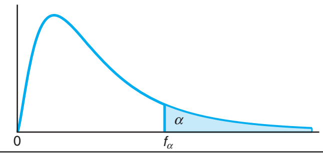


Table A.6 Critical Values of the F-Distribution

	$f_{0.05}(v_1, v_2)$								
	v_1								
v_2	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

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Table A.6 (continued) Critical Values of the F-Distribution

$f_{0.05}(v_1, v_2)$

v_2 \ v_1	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	**∞**
1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table A.6 (continued) Critical Values of the *F*-Distribution

	$f_{0.01}(v_1, v_2)$								
	v_1								
v_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Table A.6 (continued) Critical Values of the *F*-Distribution

$f_{0.01}(v_1, v_2)$

v_2 \ v_1	10	12	15	20	24	30	40	60	120	∞
1	6055.85	6106.32	6157.28	6208.73	6234.63	6260.65	6286.78	6313.03	6339.39	6365.86
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Table A.23 The Incomplete Gamma Function: $F(x;\alpha) = \int_0^x \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy$

	α									
x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0.6320	0.2640	0.0800	0.0190	0.0040	0.0010	0.0000	0.0000	0.0000	0.0000
2	0.8650	0.5940	0.3230	0.1430	0.0530	0.0170	0.0050	0.0010	0.0000	0.0000
3	0.9500	0.8010	0.5770	0.3530	0.1850	0.0840	0.0340	0.0120	0.0040	0.0010
4	0.9820	0.9080	0.7620	0.5670	0.3710	0.2150	0.1110	0.0510	0.0210	0.0080
5	0.9930	0.9600	0.8750	0.7350	0.5600	0.3840	0.2380	0.1330	0.0680	0.0320
6	0.9980	0.9830	0.9380	0.8490	0.7150	0.5540	0.3940	0.2560	0.1530	0.0840
7	0.9990	0.9930	0.9700	0.9180	0.8270	0.6990	0.5500	0.4010	0.2710	0.1700
8	1.0000	0.9970	0.9860	0.9580	0.9000	0.8090	0.6870	0.5470	0.4070	0.2830
9		0.9990	0.9940	0.9790	0.9450	0.8840	0.7930	0.6760	0.5440	0.4130
10		1.0000	0.9970	0.9900	0.9710	0.9330	0.8700	0.7800	0.6670	0.5420
11			0.9990	0.9950	0.9850	0.9620	0.9210	0.8570	0.7680	0.6590
12			1.0000	0.9980	0.9920	0.9800	0.9540	0.9110	0.8450	0.7580
13				0.9990	0.9960	0.9890	0.9740	0.9460	0.9000	0.8340
14				1.0000	0.9980	0.9940	0.9860	0.9680	0.9380	0.8910
15					0.9990	0.9970	0.9920	0.9820	0.9630	0.9300

A.24 Proof of Mean of the Hypergeometric Distribution

To find the mean of the hypergeometric distribution, we write

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} = k \sum_{x=1}^{n} \frac{(k-1)!}{(x-1)!(k-x)!} \cdot \frac{\binom{N-k}{n-x}}{\binom{N}{n}} \\
&= k \sum_{x=1}^{n} \frac{\binom{k-1}{x-1}\binom{N-k}{n-x}}{\binom{N}{n}}.
\end{aligned}
$$

Since

$$
\binom{N-k}{n-1-y} = \binom{(N-1)-(k-1)}{n-1-y} \quad \text{and} \quad \binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{N}{n}\binom{N-1}{n-1},
$$

letting $y = x - 1$, we obtain

$$
\begin{aligned}
E(X) &= k \sum_{y=0}^{n-1} \frac{\binom{k-1}{y}\binom{N-k}{n-1-y}}{\binom{N}{n}} \\
&= \frac{nk}{N} \sum_{y=0}^{n-1} \frac{\binom{k-1}{y}\binom{(N-1)-(k-1)}{n-1-y}}{\binom{N-1}{n-1}} = \frac{nk}{N},
\end{aligned}
$$

since the summation represents the total of all probabilities in a hypergeometric experiment when $N-1$ items are selected at random from $N-1$, of which $k-1$ are labeled success.