Section 1.5
Rules of Inference

Definition: A *theorem* is a *valid* logical assertion which can be proved using

- other theorems
- *axioms* (statements which are given to be true) and
- *rules of inference* (logical rules which allow the deduction of conclusions from premises).

A *lemma* (not a “lemon”) is a 'pre-theorem' or a result which is needed to prove a theorem.

A *corollary* is a 'post-theorem' or a result which follows directly from a theorem.

Rules of Inference

Many of the tautologies in Chapter 1 are rules of inference. They have the form

$$H_1 \quad H_2 \quad ..... \quad H_n \rightarrow C$$

where

H_i are called the *hypotheses*

and

C is the *conclusion*.

As a rule of inference they take the symbolic form:

$$\begin{array}{l} H_1 \\ H_2 \\ . \\ . \\ H_n \\ \quad C \end{array}$$

where means 'therefore' or 'it follows that.'

Examples:

The tautology $P \quad (P \rightarrow Q) \rightarrow Q$ becomes

$$\begin{array}{l} P \\ P \rightarrow Q \\ \quad Q \end{array}$$

This means that whenever P is true and $P \rightarrow Q$ is true we can conclude logically that Q is true.

This rule of inference is the most famous and has the name

- *modus ponens*

or

• the *law of detachment*.

Other famous rules of inference:

P $\quad P \quad Q$	Addition
$P \quad Q$ $\quad P$	Simplification
$\neg Q$ $P \to Q$ $\quad \neg P$	*Modus Tollens*
$P \to Q$ $Q \to R$ $\quad P \to R$	Hypothetical syllogism
$P \quad Q$ $\neg P$ $\quad Q$	Disjunctive syllogism
P Q $\quad P \quad Q$	Conjunction

$(P \rightarrow Q) \quad (R \rightarrow S)$
$P \quad R$
$\quad Q \quad S$

Constructive dilemma

Rules of Inference for Quantifiers

$xP(x)$ $P(c)$	Universal Instantiation (UI)
$P(x)$ $xP(x)$	Universal Generalization (UG)
$P(c)$ $xP(x)$	Existential Generalization (EG)
$xP(x)$ $P(c)$	Existential Instantiation (EI)

Note:

- In Universal Generalization, x must be arbitrary.
- In Universal Instantiation, c need not be arbitrary but often is assumed to be.

• In Existential Instantiation, c must be an element of the universe which makes P(x) true.

Example:

Every man has two legs. John Smith is a man. Therefore, John Smith has two legs.

Define the predicates:

M(x): x is a man
L(x): x has two legs
J: John Smith, a member of the universe

The argument becomes

$$1.\ \ x[M(x) \rightarrow L(x)]$$
$$2. M(J)$$
$$L(J)$$

The proof is

$1.\ \ x[M(x) \rightarrow L(x)]$	Hypothesis 1
$2. M(J) \rightarrow L(J)$	step 1 and UI
$3. M(J)$	Hypothesis 2
$4. L(J)$	steps 2 and 3 and *modus ponens*

Q. E. D.

Note: Using the rules of inference requires lots of practice.

Fallacies

Fallacies are incorrect inferences.

Some common fallacies:

• ***The Fallacy of Affirming the Consequent***

If the butler did it he has blood on his hands.
The butler had blood on his hands.
Therefore, the butler did it.

This argument has the form

$$P \to Q$$
$$Q$$
$$\quad P$$

or

$$[(P \to Q) \quad Q] \to P$$

which is not a tautology and therefore not a rule of inference!

• ***The Fallacy of Denying the Antecedent*** (or the *hypothesis*)

If the butler is nervous, he did it.
The butler is really mellow.
Therefore, the butler didn't do it.

This argument has the form

$$P \to Q$$
$$\neg P$$
$$\neg Q$$

or

$$[(P \to Q) \quad \neg P] \to \neg Q$$

which is also not a tautology and hence not a rule of inference.

• ***Begging the question*** **or** ***circular reasoning***

This occurs when we use the truth of statement being proved (or something equivalent) in the proof itself.

Example:

Conjecture: *if x^2 is even then x is even.*

Proof: If x^2 is even then $x^2 = 2k$ for some k. Then $x = 2l$ for some l. Hence, x must be even.
