Manual for SOA Exam FM/CAS Exam 2.

Chapter 7. Derivatives markets.

Section 7.5. Put options.

Extract from:

"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",

available at http://www.actexmadriver.com/

Put options

Definition 1

A **put option** *is a financial contract which gives the (***holder***)* **owner** *the right, but not the obligation, to sell a specified amount of a given security at a specified price at a specified time.*

Put options

Definition 1

A **put option** *is a financial contract which gives the (***holder***)* **owner** *the right, but not the obligation, to sell a specified amount of a given security at a specified price at a specified time.*

The put option owner exercises the option by selling the asset at the specified call price to the put writer. A put option is executed only if the put owner decides to do so. A put option owner executes a put option only when it benefits him, i.e. when the specified call price is bigger than the current (market value) spot price. Since the owner of a put option can make money if the option is exercised, put options are sold. The **owner of the put option** must pay to its counterpart for holding a put option. The price of a put option is called its **premium**.

- The (owner) buyer of a put option is called the **option put holder**. The holder of a put option is said to have a **long put position**.
- The seller of a put option is called the **option put writer**. The writer of a put is said to have a **short put position**.
- Assets used in put options are in commodities, currency exchange, stock shares and stock indices.
- A put option needs to specify the type and quality of the underlying.
- The asset used in the put option is called the **underlier** or **underlying asset**.
- The amount of the underlying asset to which the put option applies is called the **notional amount**.
- The specified price of an asset in a put option is called the **strike price**, or **exercise price**.
- A forward contract forces the buyer and seller to execute the sale. A put option is executed only if the put holder decides to do so.

- For an **European option**, the exercise of the option must occur at a certain time (the **expiration date**).
- For an **American option**, the exercise of the option must occur any time by the expiration date.
- For a **Bermudan option**, the buyer can exercise the call option during specified periods.

Unless say otherwise, we will assume that an option is an European option. European options are simpler and easier to study.

Example 1

John buys a six–month put option for 150 shares with a strike price of \$45 per share.
(i) If the price per share six months from now is \$40, John sells 150 shares to the put option writer for $(150)(45) = 6750$. *Since the market value of these 150 shares is* $(150)(40) = 6000$. *John makes (before expenses)* $6750 - 6000 = 750$ *on this contract.*
(ii) If the price per share six months from now is \$50, John does not exercise the put option.

- The put option buyer's payoff per share is

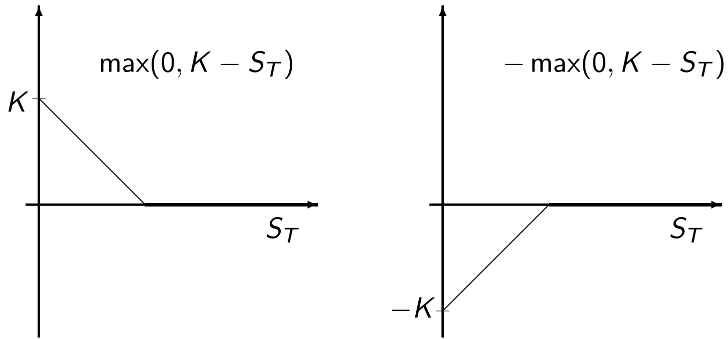
$$\max(K - S_T, 0) = \begin{cases} K - S_T & \text{if } S_T < K, \\ 0 & \text{if } S_T \geq K, \end{cases}$$

  where K is the strike price and S_T is the spot price at redemption.

- The put option writer's payoff per share is

$$-\max(K - S_T, 0) = \begin{cases} -(K - S_T) & \text{if } S_T < K, \\ 0 & \text{if } S_T \geq K, \end{cases}$$

Figure 1 shows the graph of the payoff of a put option. A put option is a zero–sum game.



Payoff for the put option holder Payoff for the put option writer

Figure 1: Payoffs of a put option

Recall:

- The put option holder's payoff is

$$\max(0, K - S_T).$$

- The put option writer's payoff is

$$-\max(0, K - S_T).$$

We get from Figure 1 that:

- The minimum payoff for the put option holder is 0. The maximum payoff for the put option holder is K.
- The minimum payoff for the put option writer is $-K$. The maximum payoff for the put option writer is 0.

	minimum payoff	maximum payoff
put option holder	0	K
put option writer	$-K$	0

Example 2

Daniel buys a 55–strike put option on XYZ stock with a nominal amount of 5000 shares. The expiration date is 6 months from now. The nominal amount of the put option is 5000 shares of XYZ stock.
(i) Calculate Daniel's payoff for the following spot prices per share at expiration: 40, 45, 55, 60, 60.
(ii) Calculate Daniel's minimum and maximum payoffs.

Example 2

Daniel buys a 55–strike put option on XYZ stock with a nominal amount of 5000 shares. The expiration date is 6 months from now. The nominal amount of the put option is 5000 shares of XYZ stock.
(i) Calculate Daniel's payoff for the following spot prices per share at expiration: 40, 45, 55, 60, 60.
(ii) Calculate Daniel's minimum and maximum payoffs.

Solution: (i) Daniel's payoff is $5000 \max(55 - S_T, 0)$. The corresponding payoffs are:

$$\begin{aligned}
&\text{if } S_T = 40, \text{ payoff} = (5000) \max(55 - 40, 0) = 75000,\\
&\text{if } S_T = 45, \text{ payoff} = (5000) \max(55 - 45, 0) = 50000,\\
&\text{if } S_T = 50, \text{ payoff} = (5000) \max(55 - 50, 0) = 25000,\\
&\text{if } S_T = 55, \text{ payoff} = (5000) \max(55 - 55, 0) = 0,\\
&\text{if } S_T = 60, \text{ payoff} = (5000) \max(55 - 60, 0) = 0.
\end{aligned}$$

(ii) Daniel's minimum payoff is zero. Daniel's maximum payoff is

Example 3

Isabella sells a 55–strike put option on XYZ stock. The expiration date is 18 months from now. The nominal amount of the put option is 10000 shares of XYZ stock.
(i) Calculate Isabella's payoff for the following spot prices per share at expiration: 40, 45, 55, 60, 60.
(ii) Calculate Isabella's minimum and maximum payoffs.

Example 3

Isabella sells a 55–strike put option on XYZ stock. The expiration date is 18 months from now. The nominal amount of the put option is 10000 shares of XYZ stock.
(i) Calculate Isabella's payoff for the following spot prices per share at expiration: 40, 45, 55, 60, 60*.*
(ii) Calculate Isabella's minimum and maximum payoffs.

Solution: (i) Isabella's payoff is $-10000\max(55 - S_T, 0)$. The corresponding payoffs are:

$$\begin{aligned}
&\text{if } S_T = 40, \text{ payoff} = -(10000)\max(55-40,0) = -150000,\\
&\text{if } S_T = 45, \text{ payoff} = -(10000)\max(55-45,0) = -100000,\\
&\text{if } S_T = 50, \text{ payoff} = -(10000)\max(55-50,0) = -50000,\\
&\text{if } S_T = 55, \text{ payoff} = -(10000)\max(55-55,0) = 0,\\
&\text{if } S_T = 60, \text{ payoff} = -(10000)\max(55-60,0) = 0.
\end{aligned}$$

(ii) Isabella's minimum payoff is $-(10000)(55) = -550000$. Isabella's maximum payoff is zero.

Let $\mathrm{Put}(K, T)$ be the premium per unit paid of a put option with strike price K and expiration time T years. Notice that $\mathrm{Put}(K, T) > 0$. Let i be the risk free annual effective rate of interest. The put option holder's profit is

$$\begin{aligned}&\max(K - S_T, 0) - \mathrm{Put}(K, T)(1+i)^T \\ =&\begin{cases} K - S_T - \mathrm{Put}(K, T)(1+i)^T & \text{if } S_T < K, \\ -\mathrm{Put}(K, T)(1+i)^T & \text{if } S_T \geq K. \end{cases}\end{aligned}$$

$\mathrm{Put}(K, T)(1+i)^T$ is the future value at time T of the purchase price. The put option writer's profit is

$$\begin{aligned}&-\max(K - S_T, 0) + \mathrm{Put}(K, T)(1+i)^T \\ =&\begin{cases} -K + S_T + \mathrm{Put}(K, T)(1+i)^T & \text{if } S_T < K, \\ \mathrm{Put}(K, T)(1+i)^T & \text{if } S_T \geq K. \end{cases}\end{aligned}$$

Figure 2 shows a graph of the put profit.

	profit
put holder	$\max(K - S_T, 0) - \mathrm{Put}(K, T)e^{rT}$
put writer	$-\max(K - S_T, 0) + \mathrm{Put}(K, T)e^{rT}$

	minimum profit	maximum profit
put holder	$-\mathrm{Put}(K, T)e^{rT}$	$K - \mathrm{Put}(K, T)e^{rT}$
put writer	$-K + \mathrm{Put}(K, T)e^{rT}$	$\mathrm{Put}(K, T)e^{rT}$

Figure 2 shows a graph of the put profit as a function of S_T.

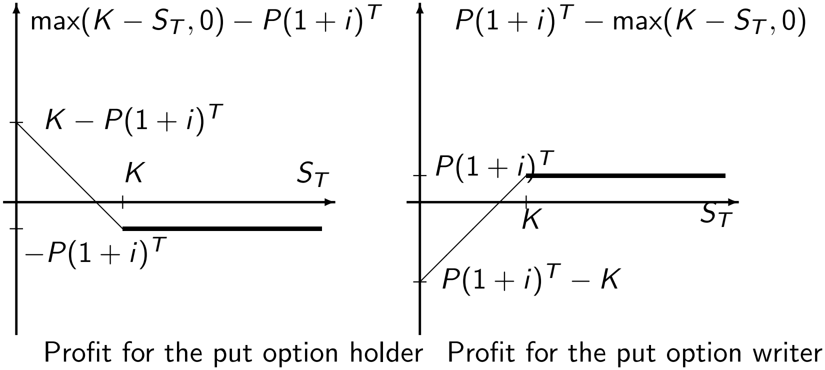


Figure 2: Profit of a put option

Notice that the put option holder's profit as a function of S_T is nonincreasing. The put option holder benefits from a decrease on the spot price. The minimum of the put option holder's profit is

$$-\mathrm{Put}(K, T)(1+i)^T.$$

The maximum of the put option's holder profit is $K - \mathrm{Put}(K, T)(1+i)^T$. If there exists no arbitrage

$$-\mathrm{Put}(K, T)(1+i)^T < 0 < K - \mathrm{Put}(K, T)(1+i)^T,$$

which is equivalent to

$$0 < \mathrm{Put}(K, T) < K(1+i)^{-T}.$$

Theorem 1

If there exists no arbitrage, then

$$\max((1+i)^{-T}K - S_0, 0) < \mathrm{Put}(K,T) < K(1+i)^{-T}.$$

Proof.

Consider the portfolio consisting of buying an asset and a put option on this asset, both for the same notional amount. The profit at expiration is

$$\begin{aligned} & S_T + \max(K - S_T, 0) - S_0(1+i)^T - \mathrm{Put}(K,T)(1+i)^T \\ = & \max(K, S_T) - (\mathrm{Put}(K,T) + S_0)(1+i)^T. \end{aligned}$$

The maximum profit is ∞. The minimum profit is $K - (\mathrm{Put}(K,T) + S_0)(1+i)^T$. If there exists no arbitrage $K - (\mathrm{Put}(K,T) + S_0)(1+i)^T < 0$, which is equivalent to $(1+i)^{-T}K - S_0 < \mathrm{Put}(K,T)$. From this bound and the bounds before the theorem, the claim follows. □

Example 4

The current price of XYZ stock is 160 per share. The annual effective interest rate is 7%. The price of a one–year European 200–strike put option for XYZ stock is \$20 per share. Find an arbitrage strategy and the minimum profit per share.

Example 4

The current price of XYZ stock is 160 per share. The annual effective interest rate is 7%. The price of a one–year European 200–strike put option for XYZ stock is \$20 per share. Find an arbitrage strategy and the minimum profit per share.

Solution: We have that

$$\mathrm{Put}(K, T) + S_0 - (1+i)^{-T}K = 20 + 160 - (200)(1.07)^{-1}$$
$$= -6.91588785 < 0.$$

The put premium is too low. Consider the portfolio consisting of buying the put and the stock, both for the same nominal amount. The profit per share is

$$\max(200 - S_T, 0) - 20(1.07) + S_T - 160(1.07) = \max(200, S_T) - 192.6.$$

The minimum profit per share is $200 - 192.6 = 7.4$.

Example 5

The current price of XYZ stock is 160 *per share. The annual effective interest rate is* 7%. *The price of a one–year European* 200*–strike put option for XYZ stock is* \$190 *per share. Find an arbitrage strategy and the minimum profit per share.*

Example 5

The current price of XYZ stock is 160 *per share. The annual effective interest rate is* 7%. *The price of a one–year European* 200*–strike put option for XYZ stock is* \$190 *per share. Find an arbitrage strategy and the minimum profit per share.*

Solution: We have that

$$(1+i)^{-T}K - \mathrm{Put}(K,T) = (200)(1.07)^{-1} - 190$$
$$= -3.08411215 < 0.$$

The put is overpriced. Consider the portfolio consisting of selling the put. The profit per share is $190(1.07) - \max(200 - S_T, 0)$. The minimum profit per share is $190(1.07) - 200 = 3.3$.

The profit of a put option holder is positive if

$$\max(K - S_T, 0) - \mathrm{Put}(K, T)(1 + i)^T > 0,$$

which is equivalent to

$$K - \mathrm{Put}(K, T)(1 + i)^T > S_T.$$

If $K - \mathrm{Put}(K, T)(1 + i)^T < S_T$, the put option holder's profit is negative.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%*.*

(i) Calculate Ashley's profit function.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(i) Calculate Ashley's profit function.

Solution: (i) Ashley's profit is

$$\begin{aligned}&(2500)(\max(85 - S_T, 0) - 4.3185816e^{0.05})\\ =&(2500)\max(85 - S_T, 0) - 11350.\end{aligned}$$

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(ii) Calculate Ashley's profit for the following spot prices at expiration: 75, 80, 85, 90, 95.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%*.*

(ii) Calculate Ashley's profit for the following spot prices at expiration: 75, 80, 85, 90, 95.

Solution: (ii) The profits corresponding to the considered spot prices are:

$$\begin{aligned}
&\text{if } S_T = 75, \text{ profit} = (2500)\max(85-75,0) - 11350 = 13650,\\
&\text{if } S_T = 80, \text{ profit} = (2500)\max(85-80,0) - 11350 = 1150,\\
&\text{if } S_T = 85, \text{ profit} = (2500)\max(85-85,0) - 11350 = -11350,\\
&\text{if } S_T = 90, \text{ profit} = (2500)\max(85-90,0) - 11350 = -11350,\\
&\text{if } S_T = 95, \text{ profit} = (2500)\max(85-95,0) - 11350 = -11350.
\end{aligned}$$

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(iii) Calculate Ashley's minimum and maximum profits.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(iii) Calculate Ashley's minimum and maximum profits.

Solution: (iii) Ashley's minimum profit is -11350. Ashley's maximum profit is $(2500)(85) - 11350 = 201150$.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(iv) Calculate the spot prices at which Ashley's profit is positive.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(iv) Calculate the spot prices at which Ashley's profit is positive.

Solution: (iv) Ashley's profit is positive if $(2500)\max(85 - S_T, 0) - 11350 > 0$, which is equivalent to $S_T < 85 - \frac{11350}{2500} = 80.46$.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(v) Calculate the spot price at expiration at which Ashley breaks even.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(v) Calculate the spot price at expiration at which Ashley breaks even.

Solution: (v) Ashley breaks even if $(2500)\max(85 - S_T, 0) - 11350 = 0$, i.e. if $S_T = 85 - \frac{11350}{2500} = 80.46$.

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%.

(vi) Calculate Ashley's annual yield in her investment for the spot prices at expiration in (ii).

Example 6

Ashley buys a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of this option is one year. The annual interest rate compounded continuously is 5%*.*

(vi) Calculate Ashley's annual yield in her investment for the spot prices at expiration in (ii).

Solution: (vi) Ashley invests $(2500)(4.3185816) = 10796.454$. Ashley's return is $(2500)\max(85 - S_T, 0)$. Let j be Ashley's annual yield. Then, $10796.454(1+j)^{0.5} = (2500)\max(85 - S_T, 0)$. Hence, $j = \left(\frac{(2500)\max(85-S_T,0)}{10796.454}\right)^2 - 1$. Therefore,

if $S_T = 75$, $j = \left(\frac{(2500)\max(85-75,0)}{10796.454}\right)^2 - 1 = 436.1888022\%$,

if $S_T = 80$, $j = \left(\frac{(2500)\max(85-80,0)}{10796.454}\right)^2 - 1 = 34.04720055\%$,

if $S_T = 85$, or 90, or 95, $j = -100\%$.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(i) Calculate William's profit function.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(i) Calculate William's profit function.

Solution: (i) William's profit is

$$
\begin{aligned}
&(2500)(4.3185816e^{0.05} - \max(85 - S_T, 0)) \\
=&11350 - (2500)\max(85 - S_T, 0).
\end{aligned}
$$

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(ii) Calculate William's profit for the following spot prices at expiration: 75, 80, 85, 90, 95.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(ii) Calculate William's profit for the following spot prices at expiration: 75, 80, 85, 90, 95.

Solution: (ii) William's profits for the considered spot prices are

$$\begin{aligned}
&\text{if } S_T = 75, \text{ profit} = 11350 - (2500)\max(85-75,0) = -13650,\\
&\text{if } S_T = 80, \text{ profit} = 11350 - (2500)\max(85-80,0) = -1150,\\
&\text{if } S_T = 85, \text{ profit} = 11350 - (2500)\max(85-85,0) = 11350,\\
&\text{if } S_T = 90, \text{ profit} = 11350 - (2500)\max(85-90,0) = 11350,\\
&\text{if } S_T = 95, \text{ profit} = 11350 - (2500)\max(85-95,0) = 11350.
\end{aligned}$$

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(iii) Calculate William's minimum and maximum profits.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(iii) Calculate William's minimum and maximum profits.

Solution: (iii) William's minimum profit is $11350 - (2500)(85) = -201150$. William's maximum profit is 11350.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(iv) Calculate the spot prices at expiration at which William makes a positive profit.

Example 7

William sells a 85–strike put option for 4.3185816 per share. The nominal amount of the put option is 2500 shares of XYZ stock. The expiration date of the option is one year. The annual interest rate compounded continuously is 5%.

(iv) Calculate the spot prices at expiration at which William makes a positive profit.

Solution: (iv) William makes a positive profit if $11350 - (2500)\max(85 - S_T, 0) > 0$, i.e. if $S_T > 85 - \frac{11350}{2500} = 80.46$.

A put option is a way to sell an asset in the future. A short forward is another way to sell an asset in the future.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for $26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is $26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of $26 per share. The exercise date is three months from now. The premium of this put option is $1.377368 per share.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for $26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is $26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of $26 per share. The exercise date is three months from now. The premium of this put option is $1.377368 per share.

(i) Find Rachel's and Dan's profits as a function of the spot price at expiration.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(i) Find Rachel's and Dan's profits as a function of the spot price at expiration.

Solution: (i) The no arbitrage price of XYZ stock is $(26)(1.055)^{0.25} = 26.35035454$. Rachel's profit is

$$(500)(26.35035 - S_T) = 13175.17727 - 500S_T.$$

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(i) Find Rachel's and Dan's profits as a function of the spot price at expiration.

Solution: (i) (continuation) Dan's profit is

$$\begin{aligned} & 500 \max(0, K - S_T) - 500\mathrm{Put}(K, T)(1+i)^T \\ = \; & 500 \max(0, 26 - S_T) - (500)(1.377368)(1.055)^{0.25} \\ = \; & 500 \max(0, 26 - S_T) - 697.9641. \end{aligned}$$

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for $26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is $26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of $26 per share. The exercise date is three months from now. The premium of this put option is $1.377368 per share.

(ii) Make a table with Rachel's and Dan's profits when the spot price at expiration is $18, $20, $22, $24, $26, $28, $30.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(ii) Make a table with Rachel's and Dan's profits when the spot price at expiration is \$18, \$20, \$22, \$24, \$26, \$28, \$30.

Solution: (ii)

Rachel's profit	4175.18	3175.18	2175.18	1175.18	175.18	−824.82	−1824.82
Dan's profit	3302.04	2302.04	1302.04	302.04	−697.96	−697.96	−697.96
Spot Price	18	20	22	24	26	28	30

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(iii) Calculate Rachel's and Dan's minimum and maximum payoffs.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(iii) Calculate Rachel's and Dan's minimum and maximum payoffs.

Solution: (iii) Rachel's minimum profit is $-\infty$. Rachel's maximum profit is 13175.18. Dan's minimum profit is -697.96. Dan's maximum profit is $(500)(26) - 697.9641 = 12302.04$.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for $26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is $26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of $26 per share. The exercise date is three months from now. The premium of this put option is $1.377368 per share.

(iv) Calculate the spot price at expiration at which Dan and Rachel make the same profit.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(iv) Calculate the spot price at expiration at which Dan and Rachel make the same profit.

Solution: (iv) Since the profits are equal for some $S_T > 26$, we solve $-697.9641 = (500)(26.35035 - S_T)$ and get $S_T = 26.35035 + 697.9641/500 = 27.74628$.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(v) Draw the graphs of the profit versus the spot price at expiration for Dan and Rachel.

Example 8

Rachel enters into a short forward contract for 500 shares of XYZ stock for \$26.35035 per share. The exercise date is three months from now. The risk free effective annual interest rate is 5.5%. The current price of XYZ stock is \$26 per share. Dan buys a put option of 500 shares of XYZ stock with a strike price of \$26 per share. The exercise date is three months from now. The premium of this put option is \$1.377368 per share.

(v) Draw the graphs of the profit versus the spot price at expiration for Dan and Rachel.

Solution: (v) The graphs of (short forward) Rachel's profit and (purchased put option) Dan's profit are in Figure 3.

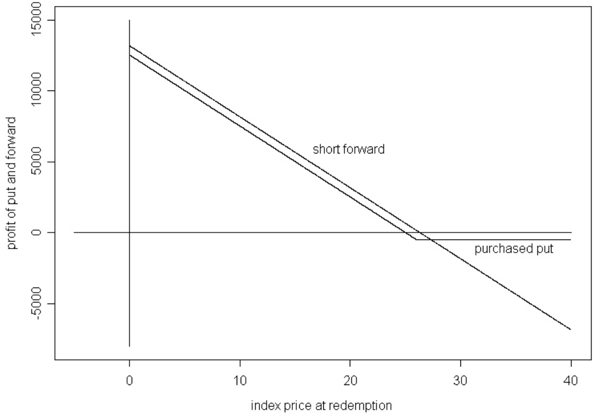


Figure 3: Profit for short forward and purchased put option.

A purchased put option reduces losses over a short forward. The profit per unit of a short forward contract is $F_{0,T} - S_T$. The minimum profit for a short forward contract is $-\infty$. The maximum profit for a short forward contract is $F_{0,T}$. The profit for the put option holder is $\max(K - S_T, 0) - \mathrm{Put}(K, T)(1+i)^T$. The minimum of the put option holder's profit is $-\mathrm{Put}(K, T)(1+i)^T$. The maximum of the put option holder's profit is $K - \mathrm{Put}(K, T)(1+i)^T$. A put option is an insured position in an asset. In return for not having large losses, the possible returns for a put option are smaller than those for a short forward.

Theorem 2

If there exists no arbitrage, then

$$\max((1+i)^{-T}(K - F_{0,t}), 0) \leq \mathrm{Put}(K, T) \leq K(1+i)^{-T}.$$

Theorem 2

If there exists no arbitrage, then

$$\max((1+i)^{-T}(K - F_{0,t}), 0) \leq \mathrm{Put}(K, T) \leq K(1+i)^{-T}.$$

Proof.

Consider the portfolio consisting of entering a long forward contract and buying a put option. The profit at expiration is

$$\begin{aligned} & S_T - F_{0,T} + \max(K - S_T, 0) - \mathrm{Put}(K, T)(1+i)^T \\ = & \max(K, S_T) - F_{0,T} - (\mathrm{Put}(K, T) + S_0)(1+i)^T. \end{aligned}$$

The maximum profit is ∞. The minimum profit is $K - F_{0,T} - (\mathrm{Put}(K, T) + S_0)(1+i)^T$. If there exists no arbitrage $K - F_{0,T} - \mathrm{Put}(K, T)(1+i)^T < 0$. The claim follows from this bound and the bounds in Theorem 1. $\square$

Example 9

The current price of a forward of corn is \$3.3 per bushel. The annual effective interest rate is 7.5%*. The price of a one–year European* 3.5*–strike put option for corn is* \$0.18 *per bushel. Find an arbitrage strategy and its minimum profit per bushel.*

Example 9

The current price of a forward of corn is \$3.3 per bushel. The annual effective interest rate is 7.5%. *The price of a one–year European* 3.5*–strike put option for corn is \$*0.18 *per bushel. Find an arbitrage strategy and its minimum profit per bushel.*

Solution: We have that

$$\begin{aligned} &\mathrm{Put}(K,T) - ((1+i)^{-T}(K - F_{0,t}) \\ =&0.18 - (1.075)(3.5 - 3.3) = -0.035 < 0. \end{aligned}$$

The put premium is too low. Consider the portfolio consisting of entering into a long forward contract and buying a put option, both for the same nominal amount. The profit is

$$\begin{aligned} &S_T - 3.3 + \max(3.5 - S_T, 0) - (0.18)(1.075) \\ =&\max(3.5, S_T) - 3.3 - (0.18)(1.075) \end{aligned}$$

The minimum profit per share is
$3.5 - 3.3 - (0.18)(1.075) = 0.0065.$

Example 10

The payoff of a purchase put option is similar to the one on a policy insurance of some asset. Suppose that you car is worth \$20000. In the case of an accident, the insurance company pays you $\max(20000 - S_T, 0)$*, where* S_T *is the price of the car after the accident. If you own stock valued at* K *and buy a put option with strike price* K*, the payoff of the put option at expiration time is* $\max(K - S_T, 0)$*. The payoff is precisely the loss in value of the stock.*

There are minor differences between these two examples. In the case of the insurance of car, usually a deductible is applied. If the deductible in your car insurance is \$500, the payment by the insurance company is $\max(20000 - 500 - S_T, 0)$*. Since cost of an accident is always positive,* $S_T < 20000$*.*

Theorem 3

If $0 < K_1 < K_2$, *then*

$$\mathrm{Put}(K_1, T) \le \mathrm{Put}(K_2, T) \le \mathrm{Put}(K_1, T) + (K_2 - K_1)e^{-rT}.$$

Proof.

We have that

$$
\begin{aligned}
&\max(K_1 - S_T, 0) \\
\leq &\max(K_2 - S_T, 0) = K_2 - K_1 + \max(K_1 - S_T, K_1 - K_2) \\
\leq &K_2 - K_1 + \max(K_1 - S_T, 0).
\end{aligned}
$$

In other words,
(i) The payoff for a K_1–strike put is smaller than or equal to the payoff for a K_2–strike put.
(ii) The payoff for a K_2–strike put is smaller than or equal to $(K_2 - K_1)$ plus the payoff for K_1–strike put.
Hence, if there exist no arbitrage, then

$$\mathrm{Put}(K_1, T) \leq \mathrm{Put}(K_2, T) \leq \mathrm{Put}(K_1, T) + (K_2 - K_1)e^{-rT}.$$

□

Example 11

The price of a one–year European 3.5*–strike put option for corn is* \$0.18 *per bushel. The price of a one–year European* 3.75*–strike put option for corn is* \$0.15 *per bushel. The annual effective interest rate is* 7.5%*. Find an arbitrage strategy and it minimum profit.*

Example 11

The price of a one–year European 3.5*–strike put option for corn is* \$0.18 *per bushel. The price of a one–year European* 3.75*–strike put option for corn is* \$0.15 *per bushel. The annual effective interest rate is* 7.5%. *Find an arbitrage strategy and it minimum profit.*

Solution: In this case, $\mathrm{Put}(K_1, T) \leq \mathrm{Put}(K_2, T)$ does not hold. We can do arbitrage by a buying a 3.75–strike put option and selling 3.5–strike put option, both for the same nominal amount. The profit per share is

$$\begin{aligned}
&\max(3.75 - S_T, 0) - \max(3.5 - S_T, 0) - (0.15 - 0.18)(1.075) \\
\geq &\max(3.75, S_T) - \max(3.5, S_T) - (0.15 - 0.18)(1.075) \\
\geq &(0.18 - 0.15)(1.075) = 0.03225.
\end{aligned}$$

Example 12

Consider two European put options on a stock, both with expiration date exactly two years from now. One put option has strike price \$85 and the other one \$95. The price of the 85–strike put is 8. The price of the 95*–strike put option is 20. The risk–free annual rate of interest compounded continuously is* 5%*. Find an arbitrage portfolio and its minimum profit.*

Example 12

Consider two European put options on a stock, both with expiration date exactly two years from now. One put option has strike price \$85 and the other one \$95. The price of the 85–strike put is 8. The price of the 95–strike put option is 20. The risk–free annual rate of interest compounded continuously is 5%. Find an arbitrage portfolio and its minimum profit.

Solution: In this case $\mathrm{Put}(K_2, T) \le \mathrm{Put}(K_1, T) + (K_2 - K_1)e^{-rT}$ does not hold. Notice that

$$\mathrm{Put}(K_1, T)+(K_2-K_1)e^{-rT} = 8+(95-85)e^{-(2)(0.05)} = 17.04837418.$$

We can do arbitrage by buying a 85–strike put option and selling a 95–strike put option. The profit per share is

$$\begin{aligned}&\max(85 - S_T, 0) - \max(95 - S_T, 0) + (20 - 8)e^{(2)(0.05)}\\ =&\max(85 - S_T, 0) - 10 - \max(85 - S_T, -10) + 13.26205102\\ \ge& - 10 + 13.26205102 = 3.26205102.\end{aligned}$$

We have the following rules of thump for price of a put option of an asset:

- Higher asset prices lead to higher put option prices.
- Higher strike prices lead to higher put option prices.
- Higher interest rates lead to higher put option prices.
- Higher expiration time lead to higher put option prices.
- Higher variation of an asset price lead to higher put option prices.

Since the put option buyer's payoff increases as the strike increases, the premium of a put option increases as the strike price increases. Hence, between two put options with different strike prices:
(i) The put option with smaller strike price has a smaller price.
(ii) If the spot price is low enough, both put options have a positive profit. The loss is bigger for put option with the higher strike price.
(iii) If the spot price is high enough, both put options have a loss. The profit is bigger for put option with the bigger strike price.

Since the strike price is paid at the expiration time, as higher the interest rate is as higher the put option price is. As higher the expiration time is as higher the put option price is.
Usually, the price of a put option is found using the Black—Scholes formula. The Black–Scholes formula for the price of a put option is

$$\mathrm{Put}(K, T) = Ke^{-rT}\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1).$$

Table 1 shows the premium of a put option for different strike prices. We have used the Black–Scholes formula with $S_0 = 26$, $t = 0.25$, $\sigma = 0.3$, $\delta = 0$ and $r = \ln(1.055)$.

Table 1:

Spot Price	22	24	26	28	30
Premium of a put option	0.1983495	0.6038701	1.377368	2.546227	4.045966

Example 13

Use the put premiums from Table 1, $i = 5.5\%$ *and* $T = 0.25$*. An investor buys a put option for 500 shares. Find the profit function for the buyer of a put option with the following strike prices: \$24, \$26, \$28.*

Example 13

Use the put premiums from Table 1, $i = 5.5\%$ and $T = 0.25$. An investor buys a put option for 500 shares. Find the profit function for the buyer of a put option with the following strike prices: \$24, \$26, \$28.

Solution: $\mathrm{Profit} = 500\max(0, K - S_T) - 500\mathrm{Put}(K, T)(1+i)^T$.

If $K = 24$, the profit is

$$\begin{aligned}&500\max(0, 24 - S_T) - (500)(0.6038701)(1.055)^{0.25}\\=&500\max(0, 24 - S_T) - 306.0036775.\end{aligned}$$

If $K = 26$, the profit is

$$\begin{aligned}&500\max(0, 26 - S_T) - (500)(1.377368)(1.055)^{0.25}\\=&500\max(0, 26 - S_T) - 697.9641.\end{aligned}$$

If $K = 28$, the profit is

$$\begin{aligned}&500\max(0, 28 - S_T) - (500)(2.546227)(1.055)^{0.25}\\=&500\max(0, 28 - S_T) - 1290.268926.\end{aligned}$$

When S_T is low, the higher the strike price is, the higher the profit is. When S_T is high, the higher the strike price is, the higher the loss is.

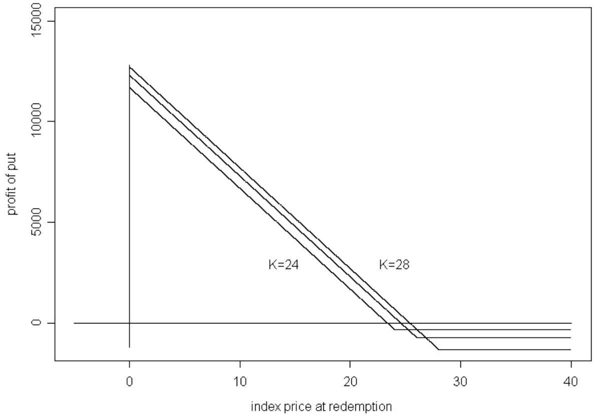


Figure 4: Profit for three puts.

- A purchased put option is in–the–money, if $S_T < K$.
- A purchased put option is out–the–money, if $S_T > K$.
- A purchased put option is at–the–money, if $S_T = K$.

If K is very small, the put option will almost certainly not be executed. Hence, if K is very small, $\mathrm{Put}(K, T) = 0$, i.e. $\lim_{K \to 0+} \mathrm{Put}(K, T) = 0$. If K is very large, the put option will almost certainly be executed. If a put is executed its profit is $K - S_T$. Hence, $\lim_{K \to \infty} \mathrm{Put}(K, T) = \infty$. As a function on K, $\mathrm{Put}(K, T)$ is an increasing function with $\lim_{K \to 0+} \mathrm{Put}(K, T) = 0$ and $\lim_{K \to \infty} \mathrm{Put}(K, T) = \infty$.

Figure 5 shows the graph of $\mathrm{Put}(K, T)$ as a function of K. $\mathrm{Put}(K, T)$ was found using the Black–Scholes formula with $T = 1$, $S_0 = 100$, $T = 1$, $\sigma = 0.25$, $r = \ln(1.06)$ and $\delta = 0.0$.

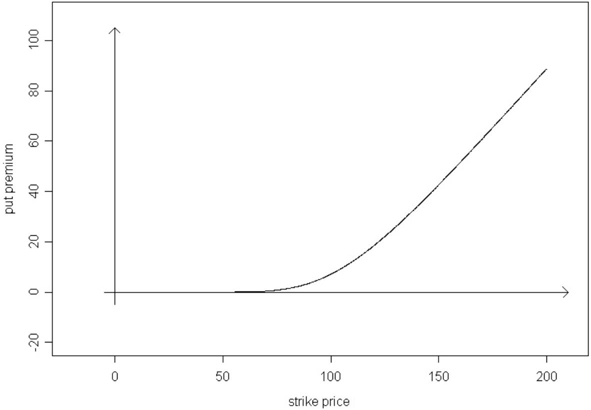


Figure 5: Graph of $\mathrm{Put}(K, T)$ as a function of K.

We have the following strategies to speculate on the change of an asset

	volatility will decrease	no volatility info	volatility will increase
price will decrease	sell a call	sell asset	buy a put
price will increase	sell a put	buy asset	buy a call

If the price of asset will decrease, we can make a profit by either selling a call, or selling asset, or buying a put. If the volatility will decrease, the chances than option is executed decrease. Hence, if the price of asset and volatility will decrease, then the preferred strategy is to sell a call.

By a similar argument, we have that:

If the price of asset will decrease and the volatility will increase, then the preferred strategy is to buy a put.

If the price of asset will increase and the volatility will decrease, then the preferred strategy is to sell a put.

If the price of asset and volatility will increase, then the preferred strategy is to buy a call.

