Manual for SOA Exam FM/CAS Exam 2.

Chapter 7. Derivatives markets.
Section 7.10. Swaps.

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Swaps

Definition 1

A **swap** *is a contract between two counterparts to exchange two similar financial quantities which behave differently.*

- The two things exchanged are called the legs of the swap.
- A common type of swap involves a commodity. Another common type of swap is an interest rate swap of a fixed interest rate in return for receiving an adjustable rate.
- Usually, one leg involves quantities that are known in advance, known as the **fixed leg**. The other involves quantities that are (uncertain) not known in advance, known as the **floating leg**.
- Usually, a swap entails the exchange of payments over time.

LIBOR.

The (**London Interbank office rate**) **LIBOR** is the most widely used reference rate for short term interest rates world–wide. The LIBOR is published daily the (British Bankers Association) BBA. It is based on rates that large international banks in London offer each other for inter–bank deposits. Rates are quoted for 1–month, 3–month, 6–month and 12–month deposits.

The following table shows the LIBOR interest rates for a loan in dollars during a week on June, 2007:

Date	1–month	3–month	6–month	9–month	12–month
6–18–2007	5.3200%	5.3600%	5.4000%	5.4400%	5.4741%
6–19–2007	5.3200%	5.3600%	5.3981%	5.4369%	5.4650%
6–20–2007	5.3200%	5.3600%	5.3931%	5.4194%	5.4387%
6–21–2007	5.3200%	5.3600%	5.3934%	5.4273%	5.4531%
6–22–2007	5.3200%	5.3600%	5.3900%	5.4200%	5.4494%

The previous rates are from `http://www.bba.org.uk/bba`.

A LIBOR loan is an adjustable loan on which the interest rate is tied to a specified Libor. The interest rate is the most recent value of the LIBOR plus a margin, subject to any adjustment cap. LIBOR is used in determining the price of interest rate futures, swaps and Eurodollars. The most important financial derivatives related to LIBOR are Eurodollar futures. Traded at the Chicago Mercantile Exchange (CME), Eurodollars are US dollars deposited at banks outside the United States, primarily in Europe. The interest rate paid on Eurodollars is largely determined by LIBOR. Eurodollar futures provide a way of betting on or hedging against future interest rate changes.

Structure of interest rates.

Let $P(0,t)$ be the price of a \$1–face value zero coupon bond maturing on date t. Notice that $\frac{1}{P(0,t)}$ is the interest factor from time zero to time t, i.e. \$1 invested at time 0 accumulates to $\frac{1}{P(0,t)}$ at time t. $P(0,t)$ is the discount factor from time zero to time t. The implied interest factor from time t_{j-1} to time t_j is $\frac{P(0,t_{j-1})}{P(0,t_j)}$. The implied forward rate from time t_{j-1} to time t_j is $r_0(t_{j-1},t_j) = \frac{P(0,t_{j-1})}{P(0,t_j)} - 1$. Let s_n be the n–year spot rate. Then, $(1+s_n)^n = \frac{1}{P(0,n)}$.

A quantity which appear later is the coupon rate R for a n–year bond with annual coupons and face value, redemption value and price all equal to one. The price of this bond is

$$1 = \sum_{j=1}^{n} RP(0,j) + P(0,n).$$

Hence,

$$R = \frac{1 - P(0,n)}{\sum_{j=1}^{n} P(0,j)}.$$

Example 1

The following table lists prices of zero–coupon \$1–face value bonds with their respective maturities:

Number of years to maturity	Price
1	\$0.956938
2	\$0.907029
3	\$0.863838
4	\$0.807217

(i) Calculate the 1–year, 2–year, 3–year, and 4–year spot rates of interest.
(ii) Calculate the 1–year, 2–year, and 3–year forward rates of interest.
(iii) Calculate the coupon rate R for a j–year bond with annual coupons whose face value, redemption value and price are all one.

Solution: (i) Note that the price of j–th bond is $P(0,j) = (1+s_j)^{-j}$. Hence, $s_j = \frac{1}{P(0,j)^{1/j}} - 1$. In particular,

$$s_1 = \frac{1}{P(0,1)} - 1 = \frac{1}{0.956938} - 1 = 4.499967135\%$$

$$s_2 = \frac{1}{P(0,2)^{1/2}} - 1 = \frac{1}{0.907029^{1/2}} - 1 = 5.000027694\%$$

$$s_3 = \frac{1}{P(0,3)^{1/3}} - 1 = \frac{1}{0.863838^{1/3}} - 1 = 4.999983734\%$$

$$s_4 = \frac{1}{P(0,4)^{1/4}} - 1 = \frac{1}{0.807217^{1/4}} - 1 = 5.499991613\%$$

Solution: (ii) To get the $j-1$ year forward rate f_j, we do $f_j = \frac{(1+s_j)^j}{(1+s_{j-1})^{j-1}} - 1 = \frac{P(0,j-1)}{P(0,j)} - 1$. We get that:

$$f_2 = \frac{0.956938}{0.907029} - 1 = 5.502470153\%,$$
$$f_3 = \frac{0.907029}{0.863838} - 1 = 4.999895814\%,$$
$$f_4 = \frac{0.863838}{0.807217} - 1 = 7.014346824\%.$$

Solution: (iii) We have that $R_j = \frac{1-P(0,j)}{\sum_{k=1}^{j} P(0,k)}$. Hence, we get that:

$$R_1 = \frac{1-0.956938}{0.956938} - 1 = 4.499978055\%,$$
$$R_2 = \frac{1-0.907029}{0.956938+0.907029} - 1 = 4.987802896\%,$$
$$R_3 = \frac{1-0.863838}{0.956938+0.907029+0.863838} - 1 = 4.991632466\%,$$
$$R_4 = \frac{1-0.807217}{0.956938+0.907029+0.863838+0.807217} - 1 = 5.453516272\%.$$

Example 2

Suppose the current LIBOR discount factors $P(0, t_j)$ are given by the table below.

LIBOR discout rates $P(0, t_j)$	0.986923	0.973921	0.961067	0.948242
time in months	3	6	9	12

Calculate the annual nominal interest rate compounded quarterly for a loan for the following maturity dates: 3, 6, 9 *and* 12 *months.*

Solution: Let $s_j^{(4)}$ be the annual nominal interest rate compounded quarterly for a loan maturing in $3j$ months, $j = 1, 2, 3, 4$. Then, $(1 + s_j^{(2)}/4)^j = \frac{1}{P(0,j/4)}$. So,

$$s_1^{(4)} = 4\left(\frac{1}{0.986923} - 1\right) = 5.300109532\%,$$

$$s_2^{(4)} = 4\left(\left(\frac{1}{0.973921}\right)^{1/2} - 1\right) = 5.320086032\%,$$

$$s_3^{(4)} = 4\left(\left(\frac{1}{0.961067}\right)^{1/3} - 1\right) = 5.330019497\%,$$

$$s_4^{(4)} = 4\left(\left(\frac{1}{0.948242}\right)^{1/4} - 1\right) = 5.350015985\%.$$

We denote by $r_{t_0}(t_1, t_2)$ to the nonannualized interest rate for the period from t_1 to t_2 using the interest rates at t_0, i.e. $1 + r_{t_0}(t_1, t_2)$ is the interest factor for the period from t_1 to t_2 using the interest rates at t_0. We denote by $P_{t_0}(t_0, t_1)$ to the price at time t_0 of a zero–coupon bond with face value \$1 and redemption time t_1. So,

$$1 + r_{t_0}(t_1, t_2) = \frac{P_{t_0}(t_0, t_1)}{P_{t_0}(t_0, t_2)}.$$

Notice that we abbreviate $P_0(0, t) = P(0, t)$.

Forward rate agreement.

Suppose that a borrower plans to take a loan of $\$L$ at time t_1, where $t_1 > 0$. He will repaid the loan at time t_2, where $t_2 > t_1$. The amount of the loan payment depends on the term structure of interest rates at time t_1. Let $r_{t_1}(t_1, t_2)$ be interest rate from t_1 to t_2 with respect to the structure of interest rates at time t_1, i.e. a zero–coupon bond with face value F and redemption time t_2 costs $\frac{F}{1+r_{t_1}(t_1,t_2)}$ at time t_1. To pay the loan, the borrower needs to pay $\$L(1 + r_{t_1}(t_1, t_2))$ at time t_2.

Since $r_{t_1}(t_1, t_2)$ is unknown at time zero, the borrower does not know how much it will have to pay for the loan. In order to hedge against increasing interest rates, the borrower can enter into a (FRA) **forward rate agreement**. A FRA is a financial contract to exchange interest payments for a notional principal on settlement date for a specified period from start date to maturity date. Usually one of the interest payments is relative to a benchmark such as the LIBOR. This is a floating interest rate, which was described in Subsection 2. The other interest payment is with respect to a fixed rate of interest. An FRA contract is settled in cash. The settlement can be made either at the beginning or at the end of the considered period, i.e. either at the borrowing time or at the time of repayment of the loan.

The two payments involved in an FRA are called legs. Both payments are made at time t_2. Usually FRA's are floating–against–fixed. One leg consists of an interest payment with respect to a floating rate. The interest payment of the floating rate leg is $Lr_{t_1}(t_1, t_2)$. The side making the floating–rate leg payment is called either the **floating–rate leg party**, or the **floating–rate side**, or the **floating–rate payer**. The interest payment of the fixed rate leg is Lr_{FRA}, where r_{FRA} is an interest rate specified in the contract. The side making the fixed–rate leg payment is called either the **fixed–rate leg party**, or the **fixed–rate side**, or the **fixed–rate payer**.

If the FRA is settled at the time of the repayment of the loan, we say that the FRA is settled in **arrears**.

Suppose that the FRA is settled at time t_2 (in arrears). The FRA is settled in two different ways:

1. If $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2)) > 0$, the fixed–rate side makes a payment of $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2))$ to the floating–rate side.

2. If $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2)) < 0$, the floating–rate side makes a payment of $L(r_{t_1}(t_1, t_2) - r_{\mathrm{FRA}})$ to the fixed–rate side.

Usually an FRA is mentioned as an exchange of (interest payments) legs. By interchanging their legs, it is meant that:
1. The floating–rate leg party makes a payment of $Lr_{t_1}(t_1, t_2)$ to its counterpart.
2. The fixed–rate leg party makes a payment of Lr_{FRA} to its counterpart.
The combination of these two payments is: the fixed–rate leg party makes a payment of $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2))$ to the floating–rate leg party. This means that if $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2))$ is a negative number, the floating–rate side makes a payment of $L(r_{t_1}(t_1, t_2) - r_{\mathrm{FRA}})$ to the fixed–rate side.

Example 3

Company A pays \$75,000 in interest payments at the end of one year. Company B pays the then–current LIBOR plus 50 basis points on a \$1,000,000 loan at the end of the year. Suppose that the two companies enter into an interest payment swap. Suppose that in one year the current LIBOR rate is 6.45%. Find which company is making a payment at the end of year and its amount.

Example 3

Company A pays \$75,000 in interest payments at the end of one year. Company B pays the then–current LIBOR plus 50 basis points on a \$1,000,000 loan at the end of the year. Suppose that the two companies enter into an interest payment swap. Suppose that in one year the current LIBOR rate is 6.45%. Find which company is making a payment at the end of year and its amount.

Solution: Company B's interest payment is $(1000000)(0.0645 + 0.0050) = 69500$. To settle the forward interest agreement, company A must make a payment of $75000 - 69500 = 5500$ to Company B.

The fixed–rate side payment is $L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2))$. Assuming that the fixed–rate side borrows L at time t_1 his total interest payment at time t_2 is

$$Lr_{t_1}(t_1, t_2) + L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2)) = Lr_{\mathrm{FRA}}.$$

A borrower can enter into an FRA as a fixed–rate side to hedge against increasing interest rates.
If the FRA is settled at time t_1 (at borrowing time), to settle the FRA the floating–rate side makes a payment of $\frac{L(r_{t_1}(t_1,t_2)-r_{\mathrm{FRA}})}{1+r_{t_1}(t_1,t_2)}$ to the fixed–rate side. This number could be negative. If $r_{\mathrm{FRA}} > r_{t_1}(t_1, t_2)$, the fixed–rate side makes a (positive) payment of $\frac{L(r_{\mathrm{FRA}}-r_{t_1}(t_1,t_2))}{1+r_{t_1}(t_1,t_2)}$ to the floating–rate side.

Since the interest factor from time t_1 to time t_2 is $1 + r_{t_1}(t_1, t_2)$, the previous payoffs are equivalent to the ones for an FRA paid in arrears. In this case, the fixed–rate side can apply the FRA payment to the principal he borrows. He takes a loan of

$$L + \frac{L(r_{\mathrm{FRA}} - r_{t_1}(t_1, t_2))}{1 + r_{t_1}(t_1, t_2)} = \frac{L(1 + r_{\mathrm{FRA}})}{1 + r_{t_1}(t_1, t_2)}$$

at time t_1. The principal of the loan at time t_2 is

$$(1 + r_{t_1}(t_1, t_2))\frac{L(1 + r_{\mathrm{FRA}})}{1 + r_{t_1}(t_1, t_2)} = L(1 + r_{\mathrm{FRA}}).$$

Again, it is like the fixed–rate side is able to borrow at the rate r_{FRA}.

Usually the floating–rate side is a market maker. The FRA agreement transfers interest rate risk from the fixed–rate side to the floating–rate side. In order to hedge this interest rate risk, the market maker could create a synthetic reverse FRA.

Suppose that the FRA is settled in arrears. The scalper buys a zero–coupon bond maturing at t_1 with face value L and short sells a zero–coupon bond with face value $\frac{LP(0,t_1)}{P(0,t_2)}$ maturing at t_2. The scalper cashflow at time zero is

$$LP(0,t_1) - \frac{LP(0,t_1)}{P(0,t_2)}P(0,t_2) = 0.$$

At time t_1, the scalper gets L from the first bond, which invests at the current interest rate. He gets $L(1 + r_{t_1}(t_1,t_2))$ at time t_2 from this investment. His total cashflow at time t_2 is

$$\begin{aligned}
&L(1 + r_{t_1}(t_1,t_2)) + L(r_{\mathrm{FRA}} - r_{t_1}(t_1,t_2)) - \frac{LP(0,t_1)}{P(0,t_2)} \\
=&L(1 + r_{t_1}(t_1,t_2)) + L(r_{\mathrm{FRA}} - r_{t_1}(t_1,t_2)) - L(1 + r_0(t_1,t_2)) \\
=&L(r_{\mathrm{FRA}} - r_0(t_1,t_2)).
\end{aligned}$$

Hence, the no arbitrage rate of an FRA is $r_0(t_1,t_2)$, which is the current nonannualized interest rate from t_1 to t_2.

Example 4

Suppose that the current spot rates are given in the following table (as annual nominal rates convertible semiannually)

spot rate	6%	7.5%
maturity (in months)	6	12

Timothy and David enter into separate forward rate agreements as fixed–rate sides for the period of time between 6 months and 12 months. Both FRA's are for a notional amount \$10000. *Timothy's FRA is settled in 12 months. David's FRA is settled in 6 months. In six months, the annual nominal interest rate compounded semiannually for a six month loan is 7%.*
(i) Find the no arbitrage six month rate for an FRA for the period of time between 6 months and 12 months.
(ii) Calculate Timothy's payoff from his FRA.
(iii) Calculate David's payoff from his FRA.

Solution: (i) The no arbitrage six month rate for a FRA for the period of time between 6 months and 12 months is

$$r_{\mathrm{FRA}} = \frac{\left(1 + \frac{0.075}{2}\right)^2}{1 + \frac{0.06}{2}} - 1 = 4.5054612\%.$$

(ii) Timothy's payoff is

$$(10000)(0.035 - 4.5054612) = 447.04612$$

(iii) David's payoff is

$$(10000)\frac{(0.035 - 4.5054612)}{1 + \frac{0.07}{2})} = 431.9286184.$$

Interest rate swaps.

An **interest rate swap** is a contract in which one party exchanges a stream of interest payments for another party's stream. Interest rate swaps are normally "**fixed–against–floating**". Interest rate swaps are valued using a notional amount. This nominal amount can change with time. We only consider constant nominal amounts. The fixed stream of payments are computed with respect to a rate determined by the contract. The floating stream of payments are determined using a benchmark, such as the LIBOR.

Suppose that a firm is interested in borrowing a large amount of money for a long time. One way to borrow is to issue bonds. Unless its credit rating is good enough, the firm may have trouble finding buyers. Lenders are unwilling to absorb long term loans from a firm with a so and so credit rating. So, the firm may have to borrow short term. Even if a company does not need to borrow short term, usually short term interest rates are lower than long term interest rates. As longer the maturity as more likely the default. Anyhow, suppose that a firm is interested in borrowing short term, but needs the cash long term. The firm takes a short term loan. At maturity, the firm pays this loan and takes another short term loan. This process will be repeated as many times as needed. Current short term rates are known. But, the short term interest rates which the firm may need to take in the future are uncertain. The firm has an interest rate risk. If short term rates increase, the company may get busted. To hedge this risk, the firm may enter into an interest rate swap, which we describe next.

Suppose that a borrower takes a loan of L paying a floating interest rate according a benchmark such as the LIBOR. Suppose that the interest is paid at times $t_1 < t_2 < \cdots < t_n$. The principal owed after each payment is L. This means that at time t_j, the borrower pays $Lr_{t_{j-1}}(t_{j-1}, t_j)$ in interest, where $1 + r_{t_{j-1}}(t_{j-1}, t_j)$ is the interest factor from time t_{j-1} to time t_j, calculated using the LIBOR at time t_{j-1}. This rate is the rate which the borrower would pay, if he borrows at time t_{j-1}, pays this loan at time t_j and takes a new loan for L at time t_j. The borrower is paying a stream of floating interest rate payments. The borrower can hedge by taking several FRA's. If the borrower would like to have a single contract, he enters into a interest rate swap.

The borrower would like to enter into an interest rate swap so that the current interest payments plus the payments to the swap will add to a fixed payment. This situation is similar to that of having several FRA's. The borrower can enter an interest rate swap with notional amount L. The borrower would like to have a fixed–rate leg in its contract:

Payment	LR	LR	$\cdots$	LR
Time	t_1	t_2	$\cdots$	t_n

where R is the **swap interest rate** in the contract. The borrower would like that its counterpart has a floating–rate leg:

Payment	$Lr_0(0, t_1)$	$Lr_{t_1}(t_1, t_2)$	$\cdots$	$Lr_{t_{n-1}}(t_{n-1}, t_n)$
Time	t_1	t_2	$\cdots$	t_n

An interest rate swap consists of an interchange of interest payments. The total outcome of this interchange is that at every time t_j the floating–leg side makes a payment of $L(r_{t_{j-1}}(t_{j-1}, t_j) - R)$ to the fixed–leg side. Again $L(r_{t_{j-1}}(t_{j-1}, t_j) - R)$ could be negative. If $L(r_{t_{j-1}}(t_{j-1}, t_j) - R) < 0$, the fixed–leg side makes a payment at each time t_j of $L(R - r_{t_{j-1}}(t_{j-1}, t_j))$ to the floating–leg side.

By exchanging legs, the borrower makes a payment at each time t_j when $L(R - r_{t_{j-1}}(t_{j-1}, t_j))$ to his counterpart. The borrower is also making interest payments of $Lr_{t_{j-1}}(t_{j-1}, t_j)$. The total borrower's interest payments add to LR. By entering a swap, a borrower is hedging against increasing interest rates.

The total borrower's cashflow is that of a company issuing bonds. Often the borrower has poor credit rating and it is unable to issue bonds. In some sense, some borrowers enter into an interest rate swap so that its counterpart issues a "bond" to them. Sometimes the borrower uses a interest rate swap to avoid to issue a fixed rate long term loan. Takers of this loan could require a higher interest to borrow.

Usually, the borrower's counterpart is a market–maker, which must hedge its interest rate risk. The (market–maker) fixed–rate payer gets a payment of $L(r_{t_{j-1}}(t_{j-1}, t_j) - R)$ at each time t_j. The fixed–rate payer profit by entering the swap is

$$\sum_{j=1}^{n} P(0, t_j) L(r_{t_{j-1}}(t_{j-1}, t_j) - R).$$

By using bonds, a market–maker can create a synthetic cashflow of payments equal to the swap payments. The cost of these bonds is its present value according with the current term structure of interest rates. Hence, if there is no arbitrage, a market–maker can arrange so that the cost of payments he receives is

$$\sum_{j=1}^{n} LP(0,t_j)(r_0(t_{j-1},t_j)-R),$$

i.e. instead of using the uncertain rates $r_{t_{j-1}}(t_{j-1},t_j)$, the scalper can use the current forward rates. Therefore, the no arbitrage swap rate is

$$R=\frac{\sum_{j=1}^{n} P(0,t_j)r_0(t_{j-1},t_j)}{\sum_{j=1}^{n} P(0,t_j)}.$$

The swap rate R when there is no arbitrage is called the **par swap rate**. Notice that the par swap rate R is a weighted average of implied forward rates $r_0(t_{j-1},t_j)$. The weights depend on the present value of a payment made at time t_j.

Using that $\frac{P(0,t_{j-1})}{P(0,t_j)} = 1 + r_0(t_{j-1}, t_j)$, we get that

$$R = \frac{\sum_{j=1}^{n} P(0, t_j) \left(\frac{P(0,t_{j-1})}{P(0,t_j)} - 1 \right)}{\sum_{j=1}^{n} P(0, t_j)}$$
$$= \frac{\sum_{j=1}^{n} P(0, t_{j-1}) - \sum_{i=1}^{n} P(0, t_j)}{\sum_{j=1}^{n} P(0, t_j)} = \frac{1 - P(0, t_n)}{\sum_{j=1}^{n} P(0, t_j)}.$$

Notice that R is the coupon rate for a bond with price, face value and redemption all equal, using the current term structure of interest rates. It is like that the floating–rate party enters the swap to use the market maker credit rating to issue a bond.

Example 5

Suppose the LIBOR discount factors $P(0, t_j)$ are given in the table below. Consider a 3–year swap with semiannual payments whose floating payments are found using the LIBOR rate compiled a semester before the payment is made. The notional amount of the swap is 10000.

LIBOR discount rates $P(0, t_j)$	0.9748	0.9492	0.9227	0.8960	0.8687	0.8413
time (months)	6	12	18	24	30	36

(i) Calculate the par swap rate.
(ii) Calculate net payment made by the fixed–rate side in 18 months if the six–month LIBOR interest rate compiled in 12 months is 2.3%.

Solution: (i) The par swap rate is

$$\begin{aligned} R &= \frac{1-P(0,t_n)}{\sum_{j=1}^{n} P(0,t_j)} \\ &= \frac{1-0.8413}{0.9748+0.9492+0.9227+0.8960+0.8687+0.8413} \\ &= 0.02910484714 = 2.910484714\%. \end{aligned}$$

(ii) The payment made by the fixed–rate side is

$$L(R-r_{t_{j-1}}(t_{j-1},t_j)) = 10000(0.02910484714-0.023) = 61.0484714.$$

Notice that the floating interest payment use the interest rates compiled one period before the payment. These interest rates are called **realized interest rates**.

Since interest rates change daily, we may be interested in the market value of a swap contract. One of the parties in the swap contract may sell/buy his position in the contract. The market value of a swap contract for the fixed–rate payer is the present value of the no arbitrage estimation of the payments which he will receive. The market value of a swap contract for the fixed–rate payer immediately after the k–the payment is

$$\sum_{j=k+1}^{n} P(t_k, t_j) L(r_{t_k}(t_{j-1}, t_j) - R).$$

If this value is positive, the fixed–rate payer has exposure to interest rates. Current interest rates are higher than when the swap was issued. The market value of a swap is the no arbitrage price to enter this contract. One of the counterparts in the contract may be interested in selling/buying his position on the contract.

Example 6

Suppose current LIBOR discount factors $P(0, t_j)$ are given by the table below. An interest rate swap has 6 payments left. The swap rate is 3.5% per period. The notional principal is two million dollars. The floating payments of this swap are the realized LIBOR interest rates.

LIBOR discount rates $P(0, t_j)$	0.9748	0.9492	0.9227	0.8960	0.8687	0.8413
time (months)	6	12	18	24	30	36

Calculate market value of this swap for the fixed–rate payer.

Solution: The market value of the swap per dollar is

$$\sum_{j=k+1}^{n} P(0,t_j)(r_{t_k}(t_{j-1},t_j)-R)$$

$$\begin{aligned}
=&0.9748\left(\frac{1}{0.9748}-1-0.035\right)+0.9492\left(\frac{0.9748}{0.9492}-1-0.035\right)\\
&+0.9227\left(\frac{0.9492}{0.9227}-1-0.035\right)+0.8960\left(\frac{0.9227}{0.8960}-1-0.035\right)\\
&+0.8687\left(\frac{0.8960}{0.8687}-1-0.035\right)+0.8413\left(\frac{0.8687}{0.8413}-1-0.035\right)\\
=&0.9748\,(0.02585145671-0.035)+0.9492\,(0.02697008007-0.035)\\
&+0.9227\,(0.02872006069-0.035)+0.8960\,(0.02979910714-0.035)\\
&+0.8687\,(0.03142626914-0.035)+0.8413\,(0.03256864377-0.035)\\
=&-0.0321445.
\end{aligned}$$

The market value of the swap is
$(2000000)(-0.0321445)=-64289.$

A **deferred swap** is a swap which begins in k periods. The swap par rate is computed as

$$R = \frac{\sum_{j=k}^{n} P(0,t_j) r_0(t_{j-1},t_j)}{\sum_{j=k}^{n} P(0,t_j)} = \frac{\sum_{j=k}^{n} P(0,t_j)\left(\frac{P(0,t_{j-1})}{P(0,t_j)} - 1\right)}{\sum_{j=k}^{n} P(0,t_j)}$$

$$= \frac{\sum_{j=k}^{n} P(0,t_{j-1}) - \sum_{j=k}^{n} P(0,t_j)}{\sum_{j=k}^{n} P(0,t_j)} = \frac{P(0,t_{k-1}) - P(0,t_n)}{\sum_{j=k}^{n} P(0,t_j)}.$$

Example 7

Suppose the current annual nominal interest rates compounded quarterly from the LIBOR are given by the table below. Two counterparts enter into a fixed against floating swap using the LIBOR rate compiled a quarter before the payment is made. The notional principal is \$50000. The times of the swap are in 12, 15 and 18 months.

$i^{(4)}$	4.5%	4.55%	4.55%	4.6%	4.6%	4.65%
maturation time in months	3	6	9	12	15	18

(i) Calculate the par swap rate.
(ii) Calculate the payment made by the fixed–rate party in 18 months if in 15 months the spot annual nominal interest rate compounded quarterly is 4.65%.

Solution: (i) The par swap rate is

$$
\begin{aligned}
R =& \frac{P(0,t_{k-1}) - P(0,t_n)}{\sum_{j=k}^{n} P(0,t_j)} \\
=& \frac{(1+0.0455/4)^{-3} - (1+0.0465/4)^{-6}}{(1+0.046/4)^{-4} + (1+0.046/4)^{-5} + (1+0.0465/4)^{-6}} \\
=& 0.01187359454 = 1.187359454\%.
\end{aligned}
$$

(ii) The payment made by the fixed–rate party is

$$L(R - r_{t_{j-1}}(t_{j-1},t_j)) = (50000)(0.01187359454 - 0.0465/4) = 12.429727.$$

The par swap rate $R = \frac{1-P(0,t_n)}{\sum_{j=1}^n P(0,t_j)}$ is a weighted average of implied forward rates. If the current interest rate does depend on the maturing time, then $P(0,t) = (1+i)^{-t}$, for some constant $i > 0$. In this case,

$$R = \frac{1-P(0,t_n)}{\sum_{j=1}^n P(0,t_j)} = \frac{1-(1+i)^{-t_n}}{\sum_{j=1}^n (1+i)^{-t_j}}.$$

If the periods in the swap have the same length, then $t_j = jh$, $1 \le j \le n$, for some $h > 0$, and

$$R = \frac{1-(1+i)^{-nh}}{\sum_{j=1}^n (1+i)^{-nj}} = \frac{1-(1+i)^{-nh}}{\frac{(1+i)^{-h}-(1+i)^{-(n+1)h}}{1-(1+i)^{-h}}}$$

$$= \frac{1}{\frac{(1+i)^{-h}}{1-(1+i)^{-h}}} = \frac{1-(1+i)^{-h}}{(1+i)^{-h}} = (1+i)^h - 1.$$

$(1+i)^h - 1$ is the effective rate for a period of length h. Notice that the assumption $P(0,t) = (1+i)^{-t}$, for some constat $i > 0$, almost never happens.

Example 8

Suppose the current annual nominal interest rates compounded quarterly from the LIBOR is 5.4% *independently of the maturity of the loan. Two counterparts enter into a fixed against floating swap using realized LIBOR rates. The times of the swap are in 3, 6 and 12 months. Calculate the par swap rate.*

Example 8

Suppose the current annual nominal interest rates compounded quarterly from the LIBOR is 5.4% *independently of the maturity of the loan. Two counterparts enter into a fixed against floating swap using realized LIBOR rates. The times of the swap are in 3, 6 and 12 months. Calculate the par swap rate.*

Solution: The par swap rate is

$$
\begin{aligned}
R &= \frac{1 - P(0, t_n)}{\sum_{j=1}^{n} P(0, t_j)} \\
&= \frac{1 - (1 + 0.054/4)^{-4}}{(1 + 0.054/4)^{-1} + (1 + 0.054/4)^{-2} + (1 + 0.054/4)^{-4}} \\
&= 0.017959328 = 1.7959328\%.
\end{aligned}
$$

Notice that in the previous problem the swap periods do not have the same length. Even if annual interest rate is constant over time, the interest payments vary over time. If the annual interest rate remains constant over time, the floating–rate payments are

$L\left((1+i)^{t_1}-1\right)$	$L\left((1+i)^{t_2-t_1}-1\right)$	$\cdots$	$L\left((1+i)^{t_n-t_{n-1}}-1\right)$
t_1	t_2	$\cdots$	t_n

Suppose that we take a loan of L at time zero. We make payments of LR at t_j, for $1 \le j \le n$. The par swap rate R is the constant periodic rate such that the final outstanding in this loan is L. Notice that the present value of the payments is $LR\sum_{j=1}^{n} P(0,t_j)$. If the final principal is $L = \frac{L - LR\sum_{j=1}^{n} P(0,t_j)}{P(0,t_n)}$, then, $R = \frac{1-P(0,t_n)}{\sum_{j=1}^{n} P(0,t_j)}$.

Commodity swaps.

A **commodity swap** is a swap where one of the legs is a commodity and the other one is cash. Hence, there are two counterparts in a swap: a party with a commodity leg and another party with a cash leg. Since the spot price of a commodity changes over time, the commodity leg is floating. Certain amount of a commodity is delivered at certain times. In some sense is like to combine several forward contracts. But, swaps are valuated considering the total deliveries. Hence, changes in interest rates change the value of a swap. Usually, the swap payment for this commodity is constant.

The commodity leg party is called the short swap side. The cash leg party is called the long swap side.

Suppose that a party would like to sell a commodity at times $0 < t_1 < t_2 < \cdots < t_n$. Suppose that the nominal amounts of the commodity are $Q_1, Q_2, \cdots, Q_n$, respectively. The commodity leg is

Amount of delivered commodity	Q_1	Q_2	$\cdots$	Q_n
Time	t_1	t_2	$\cdots$	t_n

Usually the cash leg is either

Payments	C_0	0	0	$\cdots$	0
Time	0	t_1	t_2	$\cdots$	t_n

or

Payments	C_1	C_2	$\cdots$	C_n
Time	t_1	t_2	$\cdots$	t_n

If two parties enter into a commodity swap, one will be designed the party with the commodity leg and the other the party with the cash leg. The party with the commodity leg will deliver commodity to its counterpart according with the table above. The party with the cash leg will pay cash payments to its counterpart. From a practical point of view, the party with the commodity leg is a seller. The party with the cash leg is a buyer. A commodity swap contract needs to specify the type and quality of the commodity, how to settle the contract, etc. A swap can be settled either by physical settlement or by cash settlement. If a swap is settled physically, the commodity leg side delivers the stipulated notional amount to the cash leg side, and the cash leg side pays to the commodity leg side. If a swap is cash settled, one of the parties will make a payment to the other party.

Suppose that the current forward price of this commodity with delivery in T years is $F_{0,T}$. Let $P(0,T)$ be the price of a zero–coupon with face value \$1 and expiration time T. Then, the present value of the commodity delivered is

$$\sum_{j=1}^{n} P(0,t_j) Q_j F_{0,t_j}.$$

In a **prepaid swap** the buyer makes a unique payment at time zero. If there exists no arbitrage, the price of a prepaid swap is $\sum_{j=1}^{n} P(0,t_j) Q_j F_{0,t_j}$.

Usually the cash leg consists of series of payments made at the times when the commodity is delivered. Usually each swap payment per unit of commodity is a fixed amount. Hence, the cashflow of payments is

Payment	Q_1R	Q_2R	$\cdots$	Q_nR
Time	t_1	t_2	$\cdots$	t_2

where R is the swap price per unit of commodity. The present value of the cashflow of payments is $\sum_{j=1}^{n} P(0,t_j)Q_jR$. The no arbitrage price of a swap per unit of commodity is

$$R = \frac{\sum_{j=1}^{n} P(0,t_j)Q_jF_{0,t_j}}{\sum_{j=1}^{n} P(0,t_j)Q_j}.$$

R is a weighted average of forward prices.

Suppose that at time of delivery, the buyer pays a level payment of R at each of the delivery times. Then, the present value of the cashflow of payments is $\sum_{j=1}^{n} P(0,t_j)R$. The no arbitrage level payment is

$$R = \frac{\sum_{j=1}^{n} P(0,t_j) Q_j F_{0,t_j}}{\sum_{j=1}^{n} P(0,t_j)}.$$

A commodity swap allows to lock the price of a sale. It can be used by a producer of a commodity to hedge by fixing the price that he will get in the future for this commodity. It also can be used by a manufacturer to hedge by fixing the price that he will pay in the future for a commodity. A commodity swap can be used instead of several futures/forwards. Since the price of a swap involves all the deliveries, a commodity swap involves loaning/lending somehow.

Example 9

Suppose that an airline company must buy 10,000 barrels of oil every six months, for 2 years, starting 6 months from now. The company enters into a long oil swap contract to buy this oil. The cash leg swap consists of four level payments made at the delivery times. The following table shows the annual nominal rate convertible semiannually of zero–coupon bonds maturing in $6, 12, 18, 24$ *months and the forward price of oil with delivery at those times.*

$F_{0,T}$	\$50	\$55	\$55	\$60
annual nominal rate	4.5%	5%	5%	5.5%
expiration in months	6	12	18	24

(i) Find the no arbitrage price of a prepaid swap with the commodity leg which the airline needs.

(i) Find the no arbitrage price of a prepaid swap with the commodity leg which the airline needs.

Solution: (i) The present value of the cost of oil is

$$\sum_{j=1}^{n} Q_j F_{0,t_j} P(0,t_j)$$

$$=(10000)\frac{50}{1+\frac{0.045}{2}}+(10000)\frac{55}{\left(1+\frac{0.05}{2}\right)^2}+(10000)\frac{55}{\left(1+\frac{0.05}{2}\right)^3}$$

$$+(10000)\frac{60}{\left(1+\frac{0.055}{2}\right)^4}$$

$$=488997.555+523497.9179+510729.676+538299.4402=2061524.58$$

(ii) Suppose that the airline company pays the swap by a level payment of R at each of the delivery times. Calculate R.

(ii) Suppose that the airline company pays the swap by a level payment of R at each of the delivery times. Calculate R.

Solution: (ii) We have that

$$\begin{aligned}
& \sum_{j=1}^{n} P(0, t_j) \\
= \; & \frac{1}{1+\frac{0.045}{2}} + \frac{1}{\left(1+\frac{0.05}{2}\right)^2} + \frac{1}{\left(1+\frac{0.05}{2}\right)^3} + \frac{1}{\left(1+\frac{0.055}{2}\right)^4} \\
= \; & 0.97799511 + 0.9518143962 + 0.9285994109 + 0.8971657337 \\
= \; & 3.755574651
\end{aligned}$$

and

$$R = \frac{\sum_{j=1}^{n} Q_j F_{0,t_j} P(0, t_j)}{\sum_{j=1}^{n} P(0, t_j)} = \frac{2061524.589}{37555.74651} = 548923.8747.$$

(iii) Suppose that the airline company pays the swap by unique price per barrel. Calculate the price per barrel.

(iii) Suppose that the airline company pays the swap by unique price per barrel. Calculate the price per barrel.

Solution: (iii) The price of the swap per barrel is $\frac{548923.8747}{10000} = 54.89238747$.

Example 10

Suppose that an airline company must buy 10,000 barrels of oil in two months, 12,000 barrels of oil in four months and 15,000 barrels of oil in six months. The company enters into a long oil swap. The payment of the swap will be made at the delivery times. The following table shows the annual nominal rate convertible monthly of zero–coupon bonds maturing in $6, 12, 18, 24$ *months and the forward price of oil with delivery at those times.*

Barrels of oil	\$10000	\$12000	\$15000
$F_{0,T}$	\$55	\$56	\$58
annual nominal rate	4.5%	4.55%	4.65%
expiration in months	2	4	6

(i) Suppose that the cash leg swap consists of a level payment of R at each of the delivery times. Calculate R.

Solution: (i) We have that

$$\sum_{j=1}^{n} Q_j F_{0,t_j} P(0,t_j)$$
$$=(10000)\frac{55}{\left(1+\frac{0.045}{12}\right)^2}+(12000)\frac{56}{\left(1+\frac{0.0455}{12}\right)^4}+(15000)\frac{58}{\left(1+\frac{0.0465}{12}\right)^6}$$
$$=545898.0877+661903.8839+850044.0252=2057845.997,$$

$$\begin{aligned}&\sum_{j=1}^{n} P(0,t_j)\\ =&\ \frac{1}{\left(1+\frac{0.045}{12}\right)^2}+\frac{1}{\left(1+\frac{0.0455}{12}\right)^4}+\frac{1}{\left(1+\frac{0.0465}{12}\right)^6}\\ =&\ 0.9925419775+0.9849760176+0.9770620979=2.954580093\end{aligned}$$

and

$$R=\frac{\sum_{j=1}^{n} Q_j F_{0,t_j} P(0,t_j)}{\sum_{j=1}^{n} P(0,t_j)}=\frac{2057845.997}{2.954580093}=696493.5564.$$

(ii) Suppose that the cash leg swap consists of a unique payment per barrel made at each of the delivery times. Calculate the no arbitrage swap price per barrel.

(ii) Suppose that the cash leg swap consists of a unique payment per barrel made at each of the delivery times. Calculate the no arbitrage swap price per barrel.

Solution: (ii) We have that

$$\begin{aligned}
&\sum_{j=1}^{n} P(0,t_j)Q_j\\
=\;& \frac{10000}{\left(1+\frac{0.045}{12}\right)^2} + \frac{12000}{\left(1+\frac{0.0455}{12}\right)^4} + \frac{15000}{\left(1+\frac{0.0465}{12}\right)^6}\\
=\;& 9925.419775 + 11819.712212 + 14655.931469 = 36401.06346
\end{aligned}$$

and the swap price per barrel is

$$R = \frac{\sum_{j=1}^{n} P(0,t_j)Q_j F_{0,t_j}}{\sum_{j=1}^{n} P(0,t_j)Q_j} = \frac{2057845.997}{36401.06346} = 56.53257903.$$

If a swap is cashed settled, then the commodity is valued at the current spot price. If the current value of the commodity is bigger than the value of the cash payment, then the (cash leg side) long swap pays this difference to the (commodity leg side) short swap. Reciprocally, if the current value of the commodity is smaller than the value of the cash payment, the (commodity leg side) short swap side pays this difference to the (cash leg side) long swap.

Suppose that the swap involves the sale of a commodity at times $t_1 < t_2 < \cdots < t_n$. with notional amounts of $Q_1, Q_2, \cdots, Q_n$, respectively. Suppose the swap payment is a fixed amount per unit of commodity. We saw that this amount is

$$R = \frac{\sum_{j=1}^{n} P(0, t_j) Q_j F_{0,t_j}}{\sum_{j=1}^{n} P(0, t_j) Q_j}.$$

When a swap is cash settled, the value of the commodity is found using the current spot price S_{t_j}. At time t_j the long swap pays $Q_j R - Q_j S_{t_j}$ to its counterpart. Notice that $Q_j R - Q_j S_{t_j}$ could be a negative real number. If $Q_j R - Q_j S_{t_j} < 0$, the short swap pays $Q_j S_{t_j} - Q_j R$ to its counterpart.

Changes in the forward contracts and interest rates alter the value of the swap. The market value of a swap is found using the present values of its legs using the current structure of interest rates. The market value of a long swap immediately after the settlement at time t_k is

$$\sum_{j=k+1}^{n} P(t_k, t_j) Q_j (F_{t_k, t_j} - R).$$

This is the price which an investor would pay to enter the swap as a long swap side. Immediately after the swap is undertaken the market value of the contract is

$$\sum_{j=1}^{n} Q_j (\tilde{P}(0, t_j) \tilde{F}_{0, t_j} - P(0, t_j) F_{0, t_j}),$$

where $\tilde{P}(0, t_j)$ and $\tilde{F}_{0, t_j}$ are the new market values.

Example 11

Suppose that an airline company must buy 1,000 barrels of oil every six months, for 3 years, starting 6 months from now. Instead of buying six separate long forward contracts, the company enters into a long swap contract. According with this swap the company will pay a level payment per barrel at delivery. The following table shows the annual nominal rate convertible semiannually of zero–coupon bonds maturing in $6, 12, 18, 24$ *months and the forward price of oil at those times.*

$F_{0,T}$	\$55	\$57	\$57	\$60	\$62	\$64
annual nominal rate	*5.5%*	*5.6%*	*5.65%*	*5.7%*	*5.7%*	*5.75%*
expiration in months	6	12	18	24	30	36

(i) Find the price per barrel of oil using the swap.

(i) Find the price per barrel of oil using the swap.

Solution: (i) We have that

$$\sum_{j=1}^{n} P(0,t_j)Q_jF_{0,t_j}$$
$$=(1000)\frac{55}{1+\frac{0.055}{2}}+(1000)\frac{57}{\left(1+\frac{0.056}{2}\right)^2}+(1000)\frac{57}{\left(1+\frac{0.0565}{2}\right)^3}$$
$$+(1000)\frac{60}{\left(1+\frac{0.057}{2}\right)^4}+(1000)\frac{62}{\left(1+\frac{0.057}{2}\right)^5}+(1000)\frac{64}{\left(1+\frac{0.0575}{2}\right)^6}$$
$$=54254.00740+55436.90114+54651.28637$$
$$+56698.44979+57765.24340+58747.41357=337553.3017,$$

(i) Find the price per barrel of oil using the swap.

Solution: (i)

$$
\begin{aligned}
&\sum_{j=1}^{n} P(0,t_j)Q_j \\
=&\frac{1000}{1+\frac{0.055}{2}}+\frac{1}{\left(1+\frac{0.056}{2}\right)^2}+\frac{1000}{\left(1+\frac{0.0565}{2}\right)^3} \\
&+\frac{1000}{\left(1+\frac{0.057}{2}\right)^4}+\frac{1000}{\left(1+\frac{0.057}{2}\right)^5}+\frac{1000}{\left(1+\frac{0.0575}{2}\right)^6} \\
=&986.4364982+972.5772129+958.7944977 \\
&+944.9741632+931.6974742+917.9283370=5712.408183.
\end{aligned}
$$

We have that

$$
R=\frac{\sum_{j=1}^{n} P(0,t_j)Q_jF_{0,t_j}}{\sum_{j=1}^{n} P(0,t_j)Q_j}=\frac{337553.3017}{5712.408183}=59.09124329.
$$

(ii) Suppose the swap is settled in cash. Assume that the spot rate for oil in 18 months is \$57. Calculate the payment which the airline receives.

(ii) Suppose the swap is settled in cash. Assume that the spot rate for oil in 18 months is \$57. Calculate the payment which the airline receives.

Solution: (ii) The airline gets a payment of

$$Q_j S_{t_j} - Q_j R = (1000)(57) - (1000)(59.09124329) = -2091.24329.$$

(iii) Suppose that immediately after the swap is signed up, the future prices of oil are given by the following table

$\tilde{F}_{0,T}$	\$55	\$58	\$59	\$61	\$62	\$63
expiration in months	6	12	18	24	30	36

Calculate the value of the swap for the cash leg party.

Solution: (iii) The market value of the swap for the long party is

$$
\begin{aligned}
&\sum_{j=1}^{n} P(0,t_j)(Q_j\tilde{F}_{t_0,t_j} - Q_jR)\\
=&\frac{1000}{1+\frac{0.055}{2}}(55-59.09124329)+\frac{1}{\left(1+\frac{0.056}{2}\right)^2}(58-59.09124329)\\
&+\frac{1000}{\left(1+\frac{0.0565}{2}\right)^3}(59-59.09124329)+\frac{1000}{\left(1+\frac{0.057}{2}\right)^4}(61-59.09124329)\\
&+\frac{1000}{\left(1+\frac{0.057}{2}\right)^5}(62-59.09124329)+\frac{1000}{\left(1+\frac{0.0575}{2}\right)^6}(63-59.09124329)\\
=&-4035.7517041-1061.3183576-87.4835644\\
&+1803.7257747+2710.0812797+3587.9585464=2917.211975.
\end{aligned}
$$

Suppose that immediately after the forward is signed, every future price increases by K. Then, the market value of the swap is

$$\sum_{j=1}^{n} P(0, t_j)(Q_j(F_{0,t_j} + K) - Q_j R) = \sum_{j=1}^{n} P(0, t_j) Q_j K.$$

The swap counterpart is a scalper which hedges the commodity risk resulting from the swap. The scalper has also interest rate exposure. The scalper needs to hedge changes in the price of the commodity and in interest rates.

