Manual for SOA Exam FM/CAS Exam 2.

Chapter 3. Annuities.

Section 3.5. Continuous annuities.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Continuous annuities

Annuities with length of period very small are approximately continuous annuities.

For example, the cashflows

Inflow	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm}{m}$

and

Inflow	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$	0
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm-1}{m}$	$\frac{nm}{m}$

tend to a continuous cashflow with rate $C(t) = 1$, $0 \le t \le n$, as $m \to \infty$.

Theorem 1

The present value of a continuous annuity with rate $C(t) = 1$, $0 \le t \le n$, is

$$\bar{a}_{\overline{n}|i} = \frac{1 - \nu^n}{\delta}.$$

The future value at time n of a continuous annuity with rate of one is

$$\bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\delta}.$$

Proof: We have that

$$\int_0^n \nu^t \, dt = \frac{e^{t \ln \nu}}{\ln \nu} \Big|_0^n = \frac{1 - \nu^n}{-\ln \nu} = \frac{1 - e^{-n\delta}}{\delta} = \frac{1 - \nu^n}{\delta}.$$

Recall

Theorem 2

Consider the cashflow

Inflow	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm}{m}$

Then, the present value of this cashflow is

$$a_{\overline{n}|i}^{(m)} = \frac{1-\nu^n}{i^{(m)}},$$

where $i^{(m)}$ is the nominal annual rate of interest convertible m times at year. The future value at time n of this cashflow is

$$s_{\overline{n}|i}^{(m)} = \frac{(1+i)^n-1}{i^{(m)}}.$$

Recall

Theorem 3

Consider the cashflow

Inflow	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$	0
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm-1}{m}$	$\frac{nm}{m}$

The present value of this cashflow is

$$\ddot{a}^{(m)}_{\overline{n}|i} = \frac{1-\nu^n}{d^{(m)}},$$

where $d^{(m)}$ is the nominal annual rate of discount convertible m times at year. The future value at time n of this cashflow is

$$\ddot{s}^{(m)}_{\overline{n}|i} = \frac{(1+i)^n-1}{d^{(m)}}.$$

Theorem 4

$$\bar{a}_{\overline{n}|i} = \lim_{m\to\infty} a^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|i}$$

and

$$\bar{s}_{\overline{n}|i} = \lim_{m\to\infty} s^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \ddot{s}^{(m)}_{\overline{n}|i}.$$

Proof:

$$\lim_{m\to\infty} a^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \frac{1-\nu^n}{i^{(m)}} = \frac{1-\nu^n}{\delta} = \bar{a}_{\overline{n}|i}.$$

$$\lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \frac{1-\nu^n}{d^{(m)}} = \frac{1-\nu^n}{\delta} = \bar{a}_{\overline{n}|i}.$$

Given a real number x, the integer part of x is the largest integer smaller than or equal to x, i.e. the integer k satisfying $k \leq x < k + 1$. The integer part of x is noted by $[x]$. Next theorem considers the continuous annuity with rate equal to the integer part.

Theorem 5

The present value of a continuous annuity with $C(t) = [t]$, $0 \leq t \leq n$, is

$$(I\bar{a})_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{\delta}.$$

Proof.

The present value of the continuous cashflow is

$$(I\bar{a})_{\overline{n}|i} = \int_0^n C(s)\nu^s\,ds = \sum_{j=1}^n \int_{j-1}^j j\nu^s\,ds = \sum_{j=1}^n \frac{j(\nu^j - \nu^{j-1})}{\ln \nu}$$

$$= \sum_{j=1}^n \frac{j(e^{-j\delta} - e^{-(j-1)\delta})}{-\delta} = \sum_{j=1}^n \frac{j(e^{-(j-1)\delta} - e^{-j\delta})}{\delta}$$

$$= \frac{1 + e^{-\delta} + \cdots + e^{-(n-1)\delta} - ne^{-n\delta}}{\delta}.$$

Now, $e^{-\delta} = \nu$ and

$$1 + e^{-\delta} + \cdots + e^{-(n-1)\delta} = 1 + \nu + \cdots + \nu^{n-1} = \frac{1-\nu^n}{1-\nu} = \ddot{a}_{\overline{n}|i}.$$

So, $(I\bar{a})_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{\delta}$. □

Recall

Theorem 6

The present value of the annuity

Payments	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{2}{m}$	$\cdots$	$\cdots$	$\frac{n}{m}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	1	$1+\frac{1}{m}$	$\cdots$	2	$\cdots$	$\cdots$	n

is

$$(Ia)^{(m)}_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i^{(m)}}.$$

Theorem 7

$$(I\bar{a})_{\overline{n}|i} = \lim_{m\to\infty} (Ia)^{(m)}_{\overline{n}|i} .$$

Theorem 7

$$(I\bar{a})_{\overline{n}|i} = \lim_{m\to\infty} (Ia)^{(m)}_{\overline{n}|i} .$$

Proof.

We have that

$$\lim_{m\to\infty} (Ia)^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i^{(m)}} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{\delta} = (I\bar{a})_{\overline{n}|i} .$$

□

Theorem 8

The present value of a continuous annuity with $C(t) = t$, $0 \le t \le n$, is

$$(\bar{I}\bar{a})_{\overline{n}|i} = \frac{\bar{a}_{\overline{n}|i} - n\nu^n}{\delta}.$$

Proof.

By the change of variables $x = \delta s$,

$$(\bar{I}\bar{a})_{\overline{n}|i} = \int_0^n C(s)\nu^s\,ds = \int_0^n s\nu^s\,ds = \int_0^n se^{-s\delta}\,ds$$

$$= \delta^{-2}\int_0^{n\delta} xe^{-x}\,dx = \delta^{-2}(-1-x)e^{-x}\Big|_0^{n\delta}$$

$$= \delta^{-2} - \delta^{-2}e^{-n\delta}(1+n\delta) = \frac{1-e^{-n\delta}}{\delta^2} - \frac{ne^{-n\delta}}{\delta} = \frac{\bar{a}_{\overline{n}|i} - n\nu^n}{\delta}.$$

□

Recall

Theorem 9

The present value of the annuity

Payments	0	$\frac{1}{m^2}$	$\frac{2}{m^2}$	$\frac{3}{m^2}$	$\cdots$	$\cdots$	$\frac{n}{m^2}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\frac{3}{m}$	$\cdots$	$\cdots$	n

is

$$\left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} = \frac{\ddot{a}^{(m)}_{\overline{n}|i} - n\nu^n}{i^{(m)}}.$$

Theorem 10

$$\left(\bar{I}\bar{a}\right)_{\overline{n}|i} = \lim_{m\to\infty} \left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} .$$

Theorem 10

$$(\bar{I}\bar{a})_{\overline{n}|i} = \lim_{m\to\infty} \left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} .$$

Proof.

$$\lim_{m\to\infty} \left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} = \lim_{m\to\infty} \frac{\ddot{a}^{(m)}_{\overline{n}|i} - n\nu^n}{i^{(m)}} = \lim_{m\to\infty} \frac{1-\nu^n}{i^{(m)}d^{(m)}} - \frac{n\nu^n}{i^{(m)}}$$
$$= \frac{1-\nu^n}{\delta^2} - \frac{n\nu^n}{\delta} = \frac{\bar{a}_{\overline{n}|i} - n\nu^n}{\delta} = (\bar{I}\bar{a})_{\overline{n}|i} .$$

□

