# DIOPHANTINE EQUATIONS AND BEYOND

GERD FALTINGS

## 1. INTRODUCTION

This article gives an overview of my research. My most important results was the proof of the Mordell conjecture, but this naturally led to other fields and I trie to explain how this happened.

## 2. DIOPHANTINE EQUATIONS

Diophantine problems deal with solutions of algebraic equations in rational numbers. Recall that the natural numbers $\{1, 2, 3...\}$ are obtained by simple counting, and they suffice for some purposes. However for applications one usually has to construct more complicated numbers: First one needs the zero and negative numbers $\{0, -1, ...\}$, then rational numbers $a/b$ where $a$ and $b$ are integers with $b$ different from 0. The real numbers $\mathbb{R}$ are obtained as limits of rational numbers, as for example the squareroot $\sqrt{2}$ or the number $\pi$. Finally for complex numbers $\mathbb{C}$ one has to add a squareroot $i$ of $-1$, that is they are linear combinations $a + bi$. with $a$ and $b$ real numbers. For many purposes the complex numbers suffice. For example any algebraic equation has a root in $\mathbb{C}$. However sometimes one has to replace the real numbers by different completions of the rationals. For any prime $p$ consider two rationals $a/b$ and $a'/b$ as $p$-adically close if the integer $a - a'$ is divisible by a big power of $p$. Then adding $p$-adic limits extends the rationals $\mathbb{Q}$ to the field $\mathbb{Q}_p$ of $p$-adic numbers. For example the series

$$1 + 2 - 2 + 24 -,,, = 1 - \sum_{n \geq 1} (-2)^n (1 \cdot 3 \cdot (2n - 1))/(1 \cdot ...n)$$

has a limit in $\mathbb{Q}_2$ which is equal to $\sqrt{5}$.

$\mathbb{Q}_p$ has some similarity to the reals $\mathbb{R}$ but is also quite different in certain aspects. For example it does not suffice to add one element (like $i$) to it to make it algebraically closed. Instead one has to adjoin infinitely many elements and add a further completion to extend $\mathbb{Q}_p$ to a complete algebraically closed field $\mathbb{C}_p$.

Diophantine geometry deals with solutions of algebraic equations in integers or rational numbers. Examples are Pythagorean triples (solutions in integers of $a^2 + b^2 = c^2$, or in rationals of $x^2 + y^2 = 1$), or the Fermat equation ($a^n + b^n = c^n$, or $x^n + y^n = 1$, $n \geq 3$). In general an algebraic variety is the set of common solutions of finitely many polynomial equations. Examples are $x^2 - y^2 = 1$ or $x^2 + y^3 + z^5 = 0$, but not $3^x - 2^y = 1$ (it involves nonalgebraic functions). The algebraic variety is defined over $\mathbb{Q}$ if the polynomials defining it have coefficient in $\mathbb{Q}$. It then has points (that is common solutions of the equations) in any overfield of $\mathbb{Q}$, that there are $\mathbb{Q}$-rational, $\mathbb{R}$-rational and $\mathbb{C}$-rational points. For example the algebraic variety given by $x^2 + y^2 = -1$ is defined over $\mathbb{Q}$ but has no rational points over $\mathbb{Q}$ or even over $\mathbb{R}$. On the other hand over $\mathbb{C}$ it is isomorphic to $x^2 + y^2 = 1$

(multiply $x$ and $y$ by $i$) and has many $\mathbb{C}$-rational points. This illustrates that over the complex numbers $\mathbb{C}$ many things become simpler. An algebraic variety is called smooth if the complex points form a manifold. The variety given by $x^2 + y^2 = 1$ is smooth while $y^2 - x^3 = 0$ defines a nonsmooth variety (it has a singularity at $x = y = 0$).

An important invariant of an algebraic variety is its dimension. It roughly says on how many complex parameters the $\mathbb{C}$-rational points depend. Varieties of dimension zero are finite sets. In the next case of dimension one the varieties are called curves. Such a curve has an important invariant, the genus. Rational points on curves of genus zero can be parametrised. For example rational solutions to $x^2 + y^2 = 1$ are of the form $x = (1 - t^2)/(1 + t^2), y = 2t/(1 + t^2)$. For curves of genus one Mordell showed that rational points form a finitely generated abelian group. That is they are much rarer than for genus zero, but there still may be infinitely many of them. He conjectured that for genus bigger than one the set of rational points is finite.

Important concepts in the study of diophantine equations are height and the notion of good reduction. The height of a rational number $a/b$ is the maximum of the sizes of the numerator $a$ and the denominator $b$ (assumed to be coprime). The height of an $n$-tuple $(x_1, ..., x_n)$ of rational numbers is the maximum of the height of the coordinates $x_i$. The height is important for proving finiteness theorems because it suffices to give an upper bound for the height of solutions. Of course this tends to be difficult for interesting problems.

Good reduction at a prime $p$ means in the simplest case that for a rational number $a/b$ $p$ does not divide the denominator. This can fail only for finitely many primes which are called the primes of bad reduction. Another example: An equation defines a smooth algebraic variety if some discriminant is nonzero. If the equation has integers as coefficients this discriminant is also an integer, and thus only divisible by finitely many primes. These are the primes of bad reduction for this property.

In more generality one may enlarge the rationals by adding solutions of algebraic equations to obtain algebraic numberfields. An example is the field $\mathbb{Q}(\sqrt{2})$ which consists of all linear combinations $a + b\sqrt{2}$ with rationals $a, b$. Note that this field has a symmetry by sending $a + b\sqrt{2}$ to $a - b\sqrt{2}$, quite similar to complex conjugation on $\mathbb{C}$. So if an equation has coefficients in $\mathbb{Q}$ its solutions in $\mathbb{Q}(\sqrt{2})$ admit this symmetry. Another example is obtained by adjoining to $\mathbb{Q}$ all cube roots of 2, that is $2^{1/3}, \zeta 2^{1/3}, \zeta^2 2^{1/3}$ with $\zeta = e^{2\pi i/3}$ a third root of unity. This field has degree 6 and the symmetry group consists of all permutations of the three roots. A version of it exists for all numberfields and is called the Galois-group. One of the most powerful tools in diophantine geometry consists in passing from rational points to representations of the Galois-group.

As an example we illustrate this in the case of the Fermat equation $a^n + b^n = c^n$: We want to show that it has no nontrivial solutions in integers if $n \geq 3$. It suffices to consider the cases where $n = 4$ or where $n$ is an odd prime. Then we consider the auxiliary Frey elliptic curve, that is solutions of the equation

$$y^2 = x(x - a^n)(x + b^n).$$

If we add a point at infinity the (say) complex solution ($x, y$ complex numbers) form a commutative group, that is we can define an addition of two such points. The group law is determined by the fact that the three intersection points of the curve with any straight line (in the $(x, y)$-plane) add up to zero. The height of such

an elliptic curve is given by the size of the discriminant which is $4a^n b^n c^n$, and the primes of bad reduction are the primes which divide $a$, $b$ or $c$. The $n$-division points (adding the point $n$-times to itself gives zero) form a subgroup of order $n^2$. As the group law is given by algebraic equation with coefficients in $\mathbb{Q}$ these division points lie in algebraic numberfields and thus admit an action of the Galois-group. This Galois-action has certain special properties:

Namely at the primes of bad reduction one would expect that the representation notices them (it "ramifies"). In our case (the Frey-curve for a Fermat equation) the primes of bad reduction are those dividing one of the numbers $a, b, c$. However because they occur as $n$-th powers these primes of bad reduction are not seen by the $n$-torsion points. In Wiles solution of the Fermat problem the key step is to show that this implies that the elliptic curve is "congruent modulo $n$" to an elliptic curve with no primes of bad reduction. Actually one has to work with a generalisation of elliptic curves (modular forms), but nevertheless one shows that no such object without primes of bad reduction exists, and thus the Fermat equation has no nontrivial solution.

For more complicated diophantine equations one sometimes can still associated to solutions a Frey elliptic curve. However this construction would help only if one could bound the height of this curve. To achieve this one has to solve the "abc-conjecture", one of the most important open problems in the theory.

For more general diophantine equations we cannot hope for no solutions, but only for qualitative statements like finiteness of solutions, for special types of algebraic varieties. To achieve this for curves (that is to show the Mordell conjecture) the method of Parshin-Arakelov associated to each solution a new curve and its Jacobian, which is a generalisation of the elliptic curves which we encountered for the Fermat problem. They and Szpiro could show the Mordell conjecture over function fields, which are similar to numberfields but where additional tools are available. Also Szpiro emphasized the importance of Arakelov theory which allows to carry over some techniques from function to numberfields. However one important tool (Kodaira-Spencer classes) was missing.

Quite unexpectedly this difficulty was resolved. namely associated to the auxiliary curves are their Jacobians and the Galois action on the torsion points of the Jacobians. This Galois action again has primes of bad reduction, and this set of primes is not empty but al least predetermined by the curve and not by the rational point on it. By the general theory (the Weil conjectures) there are only finitely many such representations. If we show that only finitely many points can give rise to the same representation we derive finiteness of rational points.

Now if two points give rise to the same representation the corresponding Jacobians are not the same but at least they are similar. The technical term is that they are isogenous. I could show in 1983 that in a given isogeny class the height of the Jacobian (a measure for its complexity) is bounded, and derives that there can be only finitely many Jacobians in this isogeny class. Thus there are only finitely many rational points.

The proof of the Mordell conjecture led me to further work in two fields where the necessary results could be obtained in an adhoc manner but where a fully satisfactory treatment required further work. One was the need to define heights for abelian varieties. These correspond to rational points on a certain moduli space, but to get a good theory one has to compactify that space, that is to add certain

degenerate abelian varieties. Over the complex numbers compactifications had been defined by Baily-Borel, Shimura, and Mumford, but they had no arithmetic interpretation, nor did the construction allow to bound the primes of bad reduction. However Mumford also had found a construction of degenerate abelian varieties, and I could show that Mumford's construction gives all such degenerations, and allows to define local coordinates at the boundary of the compactifications. The resulting arithmetic compactification is called the toroidal compactificationy. I wrote a book about that in collaboration with C.L.Chai.

Another development was the local theory of $p$-adic Galois representations. What was necessary for Mordell had already been done by Tate in the one dimensional case. It turned out that his method generalises to higher dimensions and yields a "p-adic Hodge decomposition". Moreover the method allowed to attack the comparison between etale and crystalline cohomology which had been conjectured by J.M.Fontaine. The whole theory now has been put on a conceptual basis by P.Scholze.

Another (in fact historically earlier) approach to diophantine equtions has been via diophantine approximation. We illustrate that for the example of Roth's theorem: If $\alpha$ is an algebraic number (that is it satisfies a polynomial equation) then $\alpha$ cannot be too well approximated by a rational numbers. Namely for any exponent $r > 2$ there are only finitely many rationals $a/b$ with

$$|\alpha - a/b| \leq 1/b^r.$$

For $r = 2$ the theory of continuous fractions gives (for real irrational numbers) infinitely many solutions. For the proof of Roth's theorem one assumes that the assertion is wrong and derives a contradiction. If the assertion is wrong there are infinitely many fractions $a/b$ satisfying the inequality. Among them the denominators $b$ can become arbitrarily large, so one can find e sequence $a_i/b_i$ with the $b_i$ rapidly increasing (which can be made precise but this requires some technicalities). Then one constructs an auxiliary polynomial $F$ in variables $T_1, ..., T_s$ of multidegree $(d_1, ..., d_s)$ (with $d_i$ approximately proportional to the invers of $log(b_i)$) which vanishes to high order at $(\alpha, ..., \alpha)$. Finally one derives a contradiction if $s$ is big enough (depending on how close $r$ is to 2) and if the $d_i$ increase rapidly enough.

Thue was the first to apply this method to diophantine geometry. He was followed by Siegel. However only Vojta succeeded much later to give a proof for the Mordell conjecture along these lines. Trying to understand his proof I developed a geometric approach which allowed to generalise it to higher dimensional varieties.

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany