Revision For Math. 203
First Mid-Term Exam
(Differential and Integral Calculus)

Q1: Determine whether the sequence $\left\{\frac{\tan^{-1} n}{n}\right\}_{n=1}^{\infty}$ is convergent or not, and if it converges find its limit.

Q2: Find the interval of convergence and the radius of convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{(x-3)^n}{n+1}$

Q3 Determine whether the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln\ n)^n}$ is absolutely convergent, conditionally convergent, or divergent

Q4: Find a power series representation for $f(x)=\ln(1+x),\ |x|<1$ and use it to calculate $\ln(1.2)$

Q5: Use the first two non-zero terms of a Maclaurin series to approximate $\int_0^{0.5} x\cos(x^2)dx$ and estimate the error in this approximation.

1

Model answer

Q1

$\because \frac{-\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$ for every n

$\Rightarrow \frac{-\pi}{2n} < \frac{\tan^{-1} n}{n} < \frac{\pi}{2n}$

$\Rightarrow \lim_{n\to\infty} \frac{-\pi}{2n} < \lim_{n\to\infty} \frac{\tan^{-1} n}{n} < \lim_{n\to\infty} \frac{\pi}{2n}$

$0 < \lim_{n\to\infty} \frac{\tan^{-1} n}{n} < 0$

$\therefore \lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0$ sandwich th.

$\Rightarrow \left\{ \frac{\tan^{-1} n}{n} \right\}$ is c'gt.

Q2 For the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{n+1}$

Apply Absolute ratio test,

let $u_n = (-1)^n \frac{1}{n+1} (x-3)^n$

$u_{n+1} = (-1)^{n+1} \frac{1}{n+2} (x-3)^{n+1}$

$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+2} \cdot \frac{n+1}{(x-3)^n} \right|$

$= |x-3|$

① If $|x-3| < 1 \Rightarrow 2 < x < 4$
$\Rightarrow$ the series is absolutely c'gt for every $x \in (2,4)$

② If $|x-3| > 1 \Rightarrow x > 4$ or $x < 2$
$\Rightarrow$ the series diverges if $x < 2$ or $x > 4$

- For $x = 4$ the series becomes $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$ which is c'gt by using AST
- For $x = 2$ the series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which is d'gt harmonic series

Hence, the interval of convergence is $(2,4]$ and the radius of convergence is $r = \frac{4-2}{2} = 1$.

#

Q3 check absolutely c'gt

let $\sum |a_n| = \sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ is absolutely c'gt.

#

2

Q4 $\because \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$, $|x|<1$

$$, \ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

$$\Rightarrow \ln(1+x) = \int_0^x [1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots]\,dt$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots \quad , |x|<1$$

$$\therefore \ln(1.2) = \ln(1+0.2)$$

$$\ln(1.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \cdots$$

$$\simeq \boxed{0.1823}$$

Q5 Maclaurin Series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\Rightarrow \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

$$\Rightarrow x\cos(x^2) = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \cdots$$

$$\Rightarrow \int_0^{0.5} x\cos(x^2)\,dx = \int_0^{0.5} [x - \frac{x^5}{2!} + \frac{x^9}{4!} - \cdots]\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^6}{2(6)} + \frac{x^{10}}{24(10)} - \cdots\right]_0^{0.5}$$

$$= \frac{(0.5)^2}{2} - \frac{(0.5)^6}{12} + \frac{(0.5)^{10}}{240} - \cdots$$

$$\Rightarrow \int_0^{0.5} x\cos(x^2)\,dx \simeq \frac{(0.5)^2}{2} - \frac{(0.5)^6}{12}$$

$$= \boxed{0.1237}$$

and $E \leq \dfrac{(0.5)^{10}}{240}$

$$\Rightarrow E \leq 4.069 \times 10^{-6}$$

#