Chapter 3: Relations

3.1 Relations (9.1 in book).

INTRODUCTION

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

DEFINITION 1

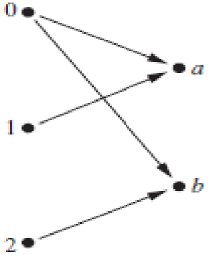
Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R. Binary relations represent relationships between the elements of two sets.

Relations

Example 1 (3 in book)

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0Ra$, but that $1 \not R b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1.



R	a	b
0	×	×
1	×	
2		×

Figure 1: Displaying the Ordered Pairs in the Relation R from Example 1.

Relations

Relations on a Set

Relations from a set A to itself are of special interest.

DEFINITION 2

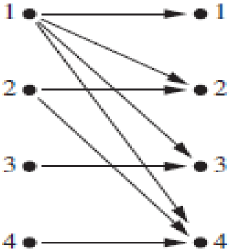
A relation on a set A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.

Example 2 (4 in book)

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$. The pairs in this relation are displayed both graphically and in tabular form in Figure 2.

Relations



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Figure 2: Displaying the Ordered Pairs in the Relation R from Example 2.

Example 3 (5 in book)

Consider these relations on the set of integers:
$R_1 = \{(a, b) | a \leq b\}$,
$R_2 = \{(a, b) | a > b\}$,
$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) | a = b\}$,
$R_5 = \{(a, b) | a = b + 1\}$,
$R_6 = \{(a, b) | a + b \leq 3\}$.
Which of these relations contain each of the pairs $(1, 1), (1, 2), (2, 1), (1, -1)$, and $(2, 2)$?

Solution: The pair $(1, 1)$ is in R_1, R_3, R_4, and R_6; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is in R_2, R_5, and R_6; $(1, -1)$ is in R_2, R_3, and R_6; and finally, $(2, 2)$ is in R_1, R_3, and R_4.

Relations

DEFINITION

- The **domain** is the set of all first elements of ordered pairs.
- The **image** is the set of all second elements of ordered pairs.

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,

$R_2 = \{(1,1),(1,2),(2,1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,

$R_6 = \{(3,4)\}$.

Gives the domain and the image of every relation.

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Consider Examples 6,7 (17,19 in book).

Example 6 (17 in book)

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
$R_1 \cap R_2 = \{(1, 1)\}$,
$R_1 - R_2 = \{(2, 2), (3, 3)\}$,
$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

Example 7 (19 in book)

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is,
$R_1 = \{(x, y)|x < y\}$ and $R_2 = \{(x, y)|x > y\}$.
What are $R_1 \cup R_2, R_1 \cap R_2, R_1 - R_2, R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y)|x \neq y\}$. In other words, the union of the "less than" relation and the "greater than" relation is the "not equals" relation. Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2) = \{(x, y)|x \neq y\}$.

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What are $R_1 \cup R_2, R_1 \cap R_2, R_1 - R_2, R_2 - R_1$, and $R_1 \oplus R_2$?
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$R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2) = \{(x, y)|x \neq y\}$.

DEFINITION 6

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation, as Examples 8 (20 in book) illustrate.

Example 8 (20 in book)

What is the composite of the relations R and S, where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(2,3)$ in R and $(3,1)$ in S produce the ordered pair $(2,1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find $S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$.

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DEFINITION 7

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots,$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

The definition shows, $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Example 9 (22 in book)

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$. Additional computation shows that R^4 is the same as R^3, so $R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. We can verify this.

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Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.
Solution: Because $R^2 = R \circ R$, we find that
$R^2 = \{(1,1),(2,1),(3,1),(4,2)\}$. Furthermore, because $R^3 = R^2 \circ R$,
$R^3 = \{(1,1),(2,1),(3,1),(4,1)\}$. Additional computation shows that R^4 is the same as R^3, so $R^4 = \{(1,1),(2,1),(3,1),(4,1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. We can verify this.

DEFINITION

Let R be a relation on the set A. An inverse relation of R is the set R^{-1} of ordered pairs obtained by interchanging the first and second elements of each pair in the original function.
If $(a, b) \in R$ then $(b, a) \in R^{-1}$

Example 9 (22 in book)

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the relations R^{-1} and $R \circ R^{-1}$.

3.2 Representing Relations and relations properties.

Representing Relations

Introduction

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Representing Relations

In other words, the zeroone matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)
The use of matrices to represent relations is illustrated in the following Examples.

Example 1

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1, a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in MR show that the pairs $(2, 1), (3, 1)$, and $(3, 2)$ belong to R. The 0s show that no other pairs belong to R.

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$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in MR show that the pairs $(2, 1), (3, 1)$, and $(3, 2)$ belong to R. The 0s show that no other pairs belong to R.

Representing Relations

Example 2

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

Example 2

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

Representing Relations

DEFINITION

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zeroone matrices. Then the join of A and B is the zeroone matrix with $(i, j)th$ entry $a_{ij} \vee b_{ij}$. The join of A and B is denoted by $A \vee B$. The meet of A and B is the matrix with $(i, j)th$ entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$. With

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} , respectively. The matrix representing the union of these relations has a 1 in the positions where either M_{R_1} or M_{R_2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both M_{R_1} and M_{R_2} have a 1. Thus, the matrices representing the union and intersection of these relations are $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ and $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$

Example 4

Suppose that the relations R_1 and R_2on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Solution: The matrices of these relations are

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Representing Relations Using Digraphs

We have shown that a relation can be represented by listing all of its ordered pairs or by using a zero-one matrix. There is another important way of representing a relation using a pictorial representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

DEFINITION 1

A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**). The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

Example 5 (7 in book)

The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is displayed in Figure 3.

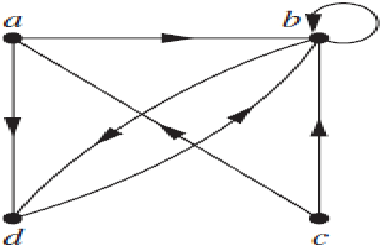

Figure 3: A Directed Graph.

Representing Relations

Example 6 (8 in book)

The directed graph of the relation
$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$ on the set
$\{1,2,3,4\}$.

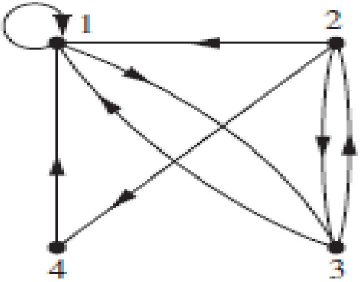

Figure 4: A Directed Graph.

Representing Relations

Example 7 (9 in book)

What are the ordered pairs in the relation R represented by the directed graph shown in Figure 5?

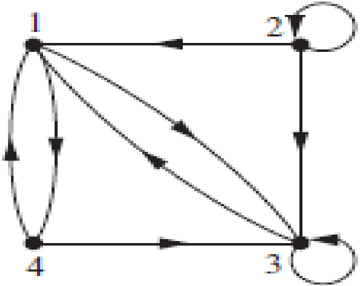

Figure 5: A Directed Graph.

The ordered pairs (x, y) in the relation are
$R = \{(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,3),(4,1),(4,3)\}$. Each of these pairs corresponds to an edge of the directed graph, with $(2,2)$ and $(3,3)$ corresponding to loops.

Properties of Relations

DEFINITION 3

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Remark

Using quantifiers we see that the relation R on the set A is reflexive if $\forall a\,((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements on the main diagonal of M_R are equal to 1, Note that the elements off the main diagonal can be either 0 or 1.

DEFINITION 4

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
A relation R on a set A such that for all $a, b \in A$, *if* $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.

Remark

Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \, \forall b \, ((a, b) \in R \rightarrow (b, a) \in R)$.
Similarly, the relation R on the set A is antisymmetric if $\forall a \, \forall b \, (((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.

In terms of the entries of M_R, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

DEFINITION 5

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark

Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a\, \forall b\, \forall c\, (((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

To prove that a relation is transitive we can use that
R is transitive if and only if $R^2 \subseteq R$

Example 3

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

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Example 4 (7,10,13 in book)

Consider the following relations on $\{1, 2, 3, 4\}$:
$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,
$R_2 = \{(1,1),(1,2),(2,1)\}$,
$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,
$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,
$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,
$R_6 = \{(3,4)\}$.

1. Which of these relations are reflexive?
2. Which of the relations are symmetric and which are antisymmetric?
3. Which of the relations are transitive?

Example 5 (8,11,14 in book)

Consider these relations on the set of integers:

$R_1 = \{(a, b) | a \leq b\}$,
$R_2 = \{(a, b) | a > b\}$,
$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) | a = b\}$,
$R_5 = \{(a, b) | a = b + 1\}$,
$R_6 = \{(a, b) | a + b \leq 3\}$.

1. Which of these relations are reflexive?
2. Which of the relations are symmetric and which are antisymmetric?
3. Which of the relations are transitive?

3.3 Equivalence Relations (9.5 in book).

Equivalence Relations

In this section we will study relations with a particular combination of properties that allows them to be used to relate objects that are similar in some way.

DEFINITION 1

A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Equivalence relations are important throughout mathematics and computer science. One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

DEFINITION 2

Two elements a and b that are related by an equivalence relation are called **equivalent**.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence Relations

Eample 1

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 3.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Eample 2

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Solution: Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation.

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Equivalence Relations

Eample 3 (6 in book)

Show that the "divides" relation is the set of positive integers in not an equivalence relation.

Solution: We show that the "divides" relation is reflexive and transitive and is not symmetric. We conclude that the "divides" relation on the set of positive integers is not an equivalence relation.

Eample 4 (7 in book)

Let R be the relation on the set of real numbers such that xRy if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

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Let R be the relation on the set of real numbers such that xRy if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Equivalence Relations

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbb{R}$. R is symmetric, for if xRy, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that yRx. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8, y = 1.9$, and $z = 1.1$, so that
$|x - y| = |2.8 - 1.9| = 0.9 < 1, |y - z| = |1.9 - 1.1| = 0.8 < 1$, but
$|x - z| = |2.81.1| = 1.7 > 1$. That is, $2.8R1.9, 1.9R1.1$, but $2.8 \not R 1.1$.

Equivalence Relations

Equivalence Classes:

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y), where x and y graduated from the same high school. Given a student x, we can form the set of all students equivalent to x with respect to R. This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

DEFINITION 3

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s | (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

Eample 5 (8 in book)

What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

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Equivalence Classes and Partitions:

THEOREM 1

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

DEFINITION

The collection of subsets A_i , $i \in I$ (where I is an index set) forms a **partition** of S if and only if $A_i \neq \emptyset$ for $i \in I$, $A_i \cap A_j = \emptyset$ when $i \neq j$.

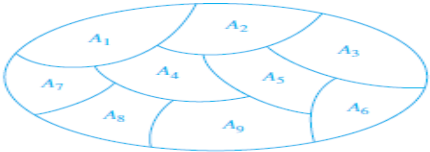

Figure 6: A Partition of a Set.

Eample 6 (12 in book)

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S.

THEOREM 2

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Equivalence Relations

Eample 7 (13 in book)

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 6.

Solution: The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition. The pairs $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)$, and $(3, 3)$ belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs $(4, 4), (4, 5), (5, 4)$, and $(5, 5)$ belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair $(6, 6)$ belongs to R because $A_3 = \{6\}$ is an equivalence class. No pair other than those listed belongs to R.

Equivalence Relations

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Solution: The subsets in the partition are the equivalence classes of R. The pair $(a,b) \in R$ if and only if a and b are in the same subset of the partition. The pairs $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)$, and $(3,3)$ belong to R because $A_1 = \{1,2,3\}$ is an equivalence class; the pairs $(4,4),(4,5),(5,4)$, and $(5,5)$ belong to R because $A_2 = \{4,5\}$ is an equivalence class; and finally the pair $(6,6)$ belongs to R because $A_3 = \{6\}$ is an equivalence class. No pair other than those listed belongs to R.

3.4 Partial Orderings (9.6 in book).

Partial Orderings

INTRODUCTION:

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

Partial Orderings

DEFINITION 1

A relation R on a set S is called a **partial ordering or partial order** if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a **partially ordered set, or poset**, and is denoted by (S, R). Members of S are called elements of the poset.

Eample 1

Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers.

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset.

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Eample 2

The divisibility relation $|$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 3.1. We see that $(\mathbb{Z}^+, |)$ is a poset. Recall that ($\mathbb{Z}^+$ denotes the set of positive integers.)

Eample 3

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset.

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Partial Orderings

Eample 4

Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if xRy, then $y \not R x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy and yRz, then xRz. However, R is not reflexive, because no person is older than himself or herself. That is, xRx is false, for all people x. It follows that R is not a partial ordering.

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DEFINITION 2

The elements a and b of a poset $(S, \leq)$ are called comparable if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, a and b are called incomparable.

Eample 5

In the poset $(\mathbb{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3|9$. The integers 5 and 7 are incomparable, because $5 \not| 7$ and $7 \not| 5$.

Remark

The adjective partial is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

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Partial Orderings

DEFINITION 3

If $(S, \leq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\leq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Eample 6

The poset $(\mathbb{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Eample 7

The poset $(\mathbb{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

Hasse Diagrams

In general, we can represent a finite poset $(S, \leq)$ using this procedure: Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops. Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x < z$ and $z < x$. Finally, arrange each edge so that its initial vertex is below its terminal vertex. Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

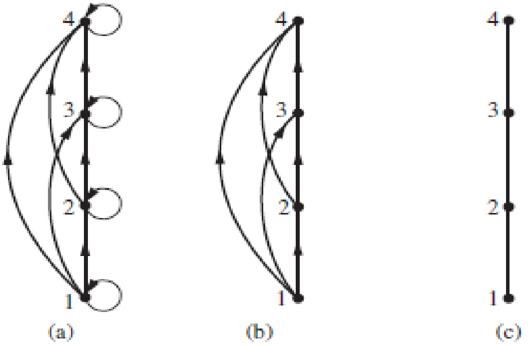

Figure 7: Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.

Partial Orderings

Eample 8 (12 in book)

Draw the Hasse diagram representing the partial ordering $\{(a, b)|a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 8(a). Remove all loops, as shown in Figure 8(b). Then delete all the edges implied by the transitive property. These are $(1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12)$, and $(3, 12)$. Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 8(c).

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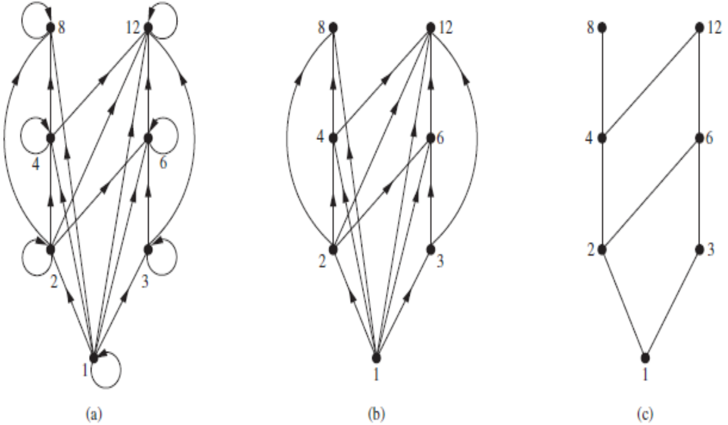

Figure 8: Constructing the Hasse Diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.

Eample 9 (13 in book)

Draw the Hasse diagram for the partial ordering $\{(A, B)|A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely,
$(\emptyset, \{a, b\}), (\emptyset, \{a, c\}), (\emptyset, \{b, c\}), (\emptyset, \{a, b, c\}), (\{a\}, \{a, b, c\}), (\{b\}, \{a, b, c\})$
and $(\{c\}, \{a, b, c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 9.

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Partial Orderings

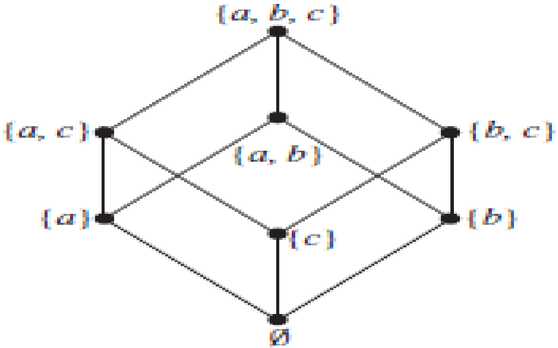

Figure 9: The Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$.

Eample 10 (14 in book)

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 10 for this poset shows that the maximal elements are $12, 20$, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.

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Partial Orderings

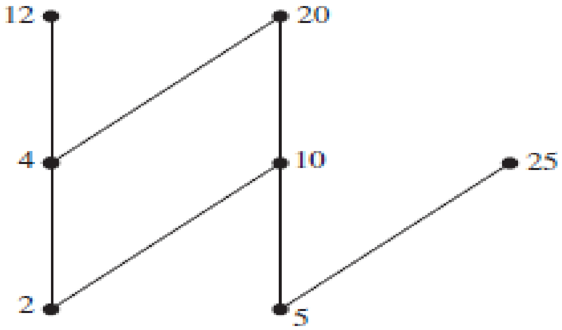

Figure 10: The Hasse Diagram of a Poset.