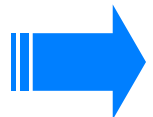


Lecture 22:
Coherent States

Phy851 Fall 2009



Summary

- Properties of the QM SHO:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

$$\lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$A = \frac{1}{\sqrt{2}} \left(\frac{X}{\lambda} + i \frac{\lambda}{\hbar} P \right)$$

$$A^\dagger = \frac{1}{\sqrt{2}} \left(\frac{X}{\lambda} - i \frac{\lambda}{\hbar} P \right)$$

$$X = \frac{\lambda}{\sqrt{2}} (A + A^\dagger)$$

$$P = -i \frac{\hbar}{\sqrt{2}\lambda} (A - A^\dagger)$$

$$H = \hbar\omega \left(A^\dagger A + \frac{1}{2} \right)$$

$$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$\psi_n(x) = \left[\sqrt{\pi} 2^n n! \lambda \right]^{-1/2} H_n(x/\lambda) e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_n(x) = \sqrt{\frac{2}{n}} \frac{x}{\lambda} \psi_{n-1}(x) - \sqrt{\frac{n-1}{n}} \psi_{n-2}(x)$$

$$\psi_0(x) = \left[\sqrt{\pi} \lambda \right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_1(x) = \left[2\sqrt{\pi} \lambda \right]^{-1/2} 2 \frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

$$\Delta X = \lambda \sqrt{n + 1/2}$$

$$\Delta P = \frac{\hbar}{\lambda} \sqrt{n + 1/2}$$



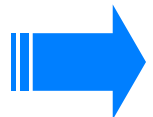
What are the 'most classical' states of the SHO?

- In HW6.4, we saw that for a minimum uncertainty wavepacket with:

$$\Delta x = \frac{\lambda_{osc}}{\sqrt{2}} \quad \lambda_{osc} = \sqrt{\frac{\hbar}{M\omega_{osc}}}$$

The uncertainties in position and momentum would remain constant.

- The interesting thing was that this was true independent of x_0 and p_0 , the initial expectation values of X and P .
- We know that other than the case $x_0=0$ and $p_0=0$, the mean position and momentum oscillate like a classical particle
- This means that for just the right initial width, the wave-packet moves around like a classical particle, but DOESN'T SPREAD at all.



'Coherent States'

- Coherent states, or as they are sometimes called 'Glauber Coherent States' are the eigenstates of the annihilation operator

$$A|\alpha\rangle = \alpha|\alpha\rangle \quad \langle\alpha|\alpha\rangle = 1$$

- Here α can be any complex number
 - i.e. there is a different coherent state for every possible choice of α
 - (Roy Glauber, Nobel Prize for Quantum Optics Theory 2005)
- These states are not really any more 'coherent' than other pure states,
 - they do maintain their coherence in the presence of dissipation somewhat more efficiently
- In QM the term 'coherence' is over-used and often abused, so do not think that it always has a precise meaning
- Glauber Coherent States are very important:
 - They are the 'most classical' states of the harmonic oscillator
 - They describe the quantum state of a laser
 - Replace the number of 'quanta' with the number of 'photons' in the laser mode
 - They describe superfluids and super-conductors



Series Solution

- Let us expand the coherent state onto energy eigenstates (i.e. number states)

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

- Plug into eigenvalue equation:

$$A|\alpha\rangle = \alpha|\alpha\rangle$$

$$A \sum_{n=0}^{\infty} c_n |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

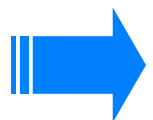
$$\sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

- Hit from left with $\langle m|$:

$$\langle m| \rightarrow \sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\sum_{n=0}^{\infty} c_n \sqrt{n} \langle m|n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n \langle m|n\rangle$$

$$c_{m+1} \sqrt{m+1} = \alpha c_m$$



Continued

$$c_{m+1} = \frac{\alpha}{\sqrt{m+1}} c_m$$

$$c_m = \frac{\alpha}{\sqrt{m}} c_{m-1}$$

- Start from: $c_0 = \mathcal{N}(\alpha)$
 - The constant $\mathcal{N}(\alpha)$ will be used at the end for normalization
- Try a few iterations:

$$c_1 = \frac{\alpha}{\sqrt{1}} c_0 = \frac{\alpha}{\sqrt{1}} \mathcal{N}(\alpha)$$

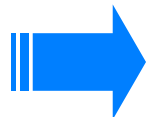
$$c_2 = \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha^2}{\sqrt{2 \cdot 1}} \mathcal{N}(\alpha)$$

$$c_3 = \frac{\alpha}{\sqrt{3}} c_2 = \frac{\alpha^3}{\sqrt{3 \cdot 2 \cdot 1}} \mathcal{N}(\alpha)$$

$$c_4 = \frac{\alpha}{\sqrt{4}} c_2 = \frac{\alpha^4}{\sqrt{4 \cdot 3 \cdot 2 \cdot 1}} \mathcal{N}(\alpha)$$

- So clearly by induction we have:

$$c_n = \frac{\alpha^n}{\sqrt{n!}} \mathcal{N}(\alpha)$$



Normalization Constant

$$c_n = \frac{\alpha^n}{\sqrt{n!}} \mathcal{N}(\alpha)$$

- So we have:

$$|\alpha\rangle = \mathcal{N}(\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

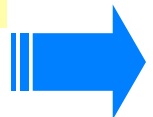
- For normalization we require:

$$\begin{aligned} 1 &= \langle \alpha | \alpha \rangle \\ &= |\mathcal{N}(\alpha)|^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{*m} \alpha^n}{\sqrt{m!n!}} \langle m | n \rangle \\ &= |\mathcal{N}(\alpha)|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\ &= |\mathcal{N}(\alpha)|^2 e^{|\alpha|^2} \end{aligned}$$

- Which gives us:

$$\mathcal{N}(\alpha) = e^{-\frac{|\alpha|^2}{2}}$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$



Orthogonality

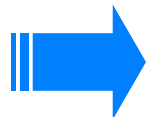
- Let us compute the inner-product of two coherent states:

$$\begin{aligned}\langle \alpha | \beta \rangle &= e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{\substack{n=0 \\ m=0}}^{\infty} \frac{\alpha^{*m} \beta^n}{\sqrt{m!n!}} \langle m | n \rangle \\ &= e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^* \beta)^n}{n!} \\ &= e^{-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha^* \beta}\end{aligned}$$

- Note that:

$$\begin{aligned}e^{-|\alpha - \beta|^2} &= e^{-(\alpha^* - \beta^*)(\alpha - \beta)} \\ &= e^{-(|\alpha|^2 + |\beta|^2 + \alpha^* \beta + \beta^* \alpha)} \\ &= |\langle \alpha | \beta \rangle|^2\end{aligned}$$

- So coherent states are *NOT* orthogonal
 - Does this contradict our earlier results regarding the orthogonality of eigenstates?



Expectation Values of Position Operator

- Lets look at the shape of the coherent state wavepacket

- Let $\psi_\alpha(x) = \langle x|\alpha\rangle$

$$\langle X \rangle = \int dx \psi_\alpha^*(x) x \psi_\alpha(x)$$

- Better to avoid these integrals, instead lets try using A and A^\dagger :

$$\langle X \rangle = \langle \alpha | \frac{\lambda}{\sqrt{2}} (A + A^\dagger) | \alpha \rangle$$

- Recall the definition of $|\alpha\rangle$:

$$A|\alpha\rangle = \alpha|\alpha\rangle \quad \langle \alpha|A^\dagger = \alpha^*\langle \alpha|$$

$$\langle X \rangle = \frac{\lambda}{\sqrt{2}} (\langle \alpha|A|\alpha\rangle + \langle \alpha|A^\dagger|\alpha\rangle)$$

$$= \frac{\lambda}{\sqrt{2}} (\alpha\langle \alpha|\alpha\rangle + \alpha^*\langle \alpha|\alpha\rangle)$$

$$= \frac{\lambda}{\sqrt{2}} (\alpha + \alpha^*)$$

$$\langle X \rangle = \sqrt{2}\lambda \text{Re}\{\alpha\}$$



Expectation Value of Momentum Operator

- We can follow the same procedure for the momentum:

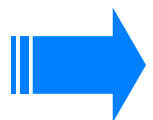
$$\begin{aligned}\langle P \rangle &= -i \frac{\hbar}{\sqrt{2}\lambda} \langle \alpha | (A - A^\dagger) | \alpha \rangle \\ &= \frac{\sqrt{2}\hbar}{2i\lambda} (\langle \alpha | A | \alpha \rangle - \langle \alpha | A^\dagger | \alpha \rangle) \\ &= \frac{\sqrt{2}\hbar}{2i\lambda} (\alpha - \alpha^*)\end{aligned}$$

$$\langle P \rangle = \frac{\sqrt{2}\hbar}{\lambda} \text{Im}\{\alpha\}$$

$$\langle X \rangle = \sqrt{2}\lambda \text{Re}\{\alpha\}$$

- Not surprisingly, this gives:

$$\alpha = \frac{1}{\sqrt{2}} \left(\frac{1}{\lambda} \langle X \rangle + i \frac{\lambda}{\hbar} \langle P \rangle \right)$$



Variance in Position

- Now let us compute the spread in x :

$$\begin{aligned}\langle X^2 \rangle &= \langle \alpha | \frac{\lambda^2}{2} (A + A^\dagger)^2 | \alpha \rangle \\ &= \langle \alpha | \frac{\lambda^2}{2} (A^2 + AA^\dagger + A^\dagger A + A^\dagger A^\dagger) | \alpha \rangle\end{aligned}$$

- Put all of the A 's on the right and the A^\dagger 's on the left:
 - This is called 'Normal Ordering'

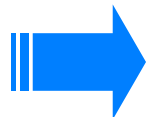
$$\begin{aligned}&= \langle \alpha | \frac{\lambda^2}{2} (A^2 + 2A^\dagger A + 1 + A^\dagger A^\dagger) | \alpha \rangle \\ &= \frac{\lambda^2}{2} (\alpha^2 + 2\alpha^* \alpha + 1 + \alpha^{*2}) \\ &= \frac{\lambda^2}{2} ((\alpha + \alpha^*)^2 + 1)\end{aligned}$$

$$\langle X \rangle = \frac{\lambda}{\sqrt{2}} (\alpha + \alpha^*)$$

$$\langle X^2 \rangle = \langle X \rangle^2 + \frac{\lambda^2}{2}$$

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\lambda}{\sqrt{2}}$$

Exactly the same variance as the ground state $|n=0\rangle$



Momentum Variance

- Similarly, we have:

$$\begin{aligned}\langle P^2 \rangle &= -\frac{\hbar^2}{2\lambda^2} \langle \alpha | (A - A^\dagger)^2 | \alpha \rangle \\ &= -\frac{\hbar^2}{2\lambda^2} \langle \alpha | (AA - AA^\dagger - A^\dagger A + A^\dagger A^\dagger) | \alpha \rangle\end{aligned}$$

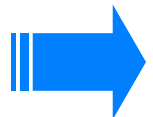
- Normal ordering gives:

$$\begin{aligned}\langle P^2 \rangle &= -\frac{\hbar^2}{2\lambda^2} \langle \alpha | (AA - 2A^\dagger A - 1 + A^\dagger A^\dagger) | \alpha \rangle \\ &= -\frac{\hbar^2}{2\lambda^2} \langle \alpha | (AA - 2A^\dagger A - 1 + A^\dagger A^\dagger) | \alpha \rangle \\ &= -\frac{\hbar^2}{2\lambda^2} (\alpha^2 - 2\alpha^* \alpha + \alpha^{*2} - 1) \\ &= -\frac{\hbar^2}{2\lambda^2} ((\alpha - \alpha^*)^2 - 1)\end{aligned}$$

$$\langle P \rangle = \frac{\sqrt{2}\hbar}{2i\lambda} (\alpha - \alpha^*)$$

$$\langle P^2 \rangle = \langle P \rangle^2 + \frac{\hbar^2}{2\lambda^2}$$

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\sqrt{2}\lambda}$$



Minimum Uncertainty States

- Let us check what Heisenberg Uncertainty Relation says about coherent states:

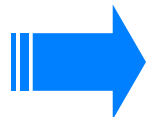
$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\lambda}{\sqrt{2}}$$

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\sqrt{2}\lambda}$$

$$\Delta X \Delta P = \frac{\lambda}{\sqrt{2}} \frac{\hbar}{\sqrt{2}\lambda}$$

$$\Delta X \Delta P = \frac{\hbar}{2}$$

- So we see that all coherent states (meaning no matter what complex value α takes on) are *Minimum Uncertainty States*
 - This is one of the reasons we say they are 'most classical'



Time Evolution

- We can easily determine the time evolution of the coherent states, since we have already expanded onto the Energy Eigenstates:

- Let

$$|\psi(t=0)\rangle = |\alpha_0\rangle$$

- Thus we have:

$$|\psi(0)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$$

$$|\psi(t)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega(n+1/2)t} |n\rangle$$

$$= e^{-i\omega t/2} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega n t} |n\rangle$$

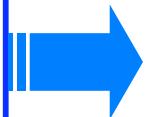
$$= e^{-i\omega t/2} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_0 e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

- Let

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

$$\psi(t) = |\alpha(t)\rangle$$

By this we mean it remains in a coherent state, but the value of the parameter α changes in time



Why 'most classical'?

- What we have learned:
 - Coherent states remain coherent states as time evolves, but the parameter α changes in time as

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

- This means they remain a minimum uncertainty state at all time
- The momentum and position variances are the same as the $n=0$ Energy eigenstate
- Recall that:

$$\langle X \rangle = \sqrt{2\lambda} \operatorname{Re}\{\alpha\}$$

$$\langle P \rangle = \frac{\sqrt{2\hbar}}{\lambda} \operatorname{Im}\{\alpha\}$$

- So we can see that:

$$\alpha_0 = \frac{1}{\sqrt{2}} \left(\frac{x_0}{\lambda} + i \frac{\lambda}{\hbar} p_0 \right) \quad \begin{aligned} x_0 &= \langle \alpha(t) | X | \alpha(t) \rangle \\ p_0 &= \langle \alpha(t) | P | \alpha(t) \rangle \end{aligned}$$

- We already know that $\langle X \rangle$ and $\langle P \rangle$ behave as classical particle in the Harmonic Oscillator, for any initial state.

$$x(t) = x_0 \cos(\omega t) + \frac{p_0}{\omega} \sin(\omega t) \quad p(t) = p_0 \cos(\omega t) - \omega x_0 \sin(\omega t)$$



Conclusions

- The Coherent State wavefunction looks exactly like ground state, but shifted in momentum and position. It then moves as a classical particle, while keeping its shape fixed.
 - Note: the coherent state is also called a 'Displaced Ground State'

