Lecture 22: Coherent States

Phy851 Fall 2009

Summary

- Properties of the QM SHO:

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 \qquad \lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$A = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} + i\frac{\lambda}{\hbar}P\right) \qquad A^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right)$$

$$X = \frac{\lambda}{\sqrt{2}}\left(A + A^\dagger\right) \qquad P = -i\frac{\hbar}{\sqrt{2}\lambda}\left(A - A^\dagger\right)$$

$$H = \hbar\omega\left(A^\dagger A + \frac{1}{2}\right)$$

$$H\left|n\right\rangle = \hbar\omega(n + 1/2)\left|n\right\rangle$$

$$A\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \qquad A^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \qquad \left|n\right\rangle = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}}\left|0\right\rangle$$

$$\psi_n(x) = \left[\sqrt{\pi}\,2^n\, n!\,\lambda\right]^{-1/2} H_n\left(x/\lambda\right) e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_n(x) = \sqrt{\frac{2}{n}}\frac{x}{\lambda}\psi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\psi_{n-2}(x)$$

$$\psi_0(x) = \left[\sqrt{\pi}\lambda\right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}} \qquad \psi_1(x) = \left[2\sqrt{\pi}\lambda\right]^{-1/2} 2\frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

$$\Delta X = \lambda\sqrt{n + 1/2} \qquad \Delta P = \frac{\hbar}{\lambda}\sqrt{n + 1/2}$$

What are the `most classical' states of the SHO?

- In HW6.4, we saw that for a minimum uncertainty wavepacket with:

$$\Delta x = \frac{\lambda_{osc}}{\sqrt{2}} \qquad \lambda_{osc} = \sqrt{\frac{\hbar}{M\omega_{osc}}}$$

  The uncertainties in position and momentum would remain constant.

- The interesting thing was that this was true independent of x_0 and p_0, the initial expectation values of X and P.

- We know that other than the case $x_0=0$ and $p_0=0$, the mean position and momentum oscillate like a classical particle

- This means that for just the right initial width, the wave-packet moves around like a classical particle, but DOESN'T SPREAD at all.

‘Coherent States’

- Coherent states, or as they are sometimes called ‘Glauber Coherent States’ are the eigenstates of the annihilation operator

$$A|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle\alpha|\alpha\rangle = 1$$

  - Here α can be any complex number
  - i.e. there is a different coherent state for every possible choice of α
  - (Roy Glauber, Nobel Prize for Quantum Optics Theory 2005)
- These states are not really any more ‘coherent’ then other pure states,
  - they do maintain their coherence in the presence of dissipation somewhat more efficiently

- In QM the term ‘coherence’ is over-used and often abused, so do not think that it always has a precise meaning

- Glauber Coherent States are very important:
  - They are the ‘most classical’ states of the harmonic oscillator
  - They describe the quantum state of a laser
    - Replace the number of ‘quanta’ with the number of ‘photons’ in the laser mode
  - They describe superfluids and super-conductors

Series Solution

- Let us expand the coherent state onto energy eigenstates (i.e. number states)

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

- Plug into eigenvalue equation:

$$A|\alpha\rangle = \alpha|\alpha\rangle$$

$$A\sum_{n=0}^{\infty} c_n |n\rangle = \alpha\sum_{n=0}^{\infty} c_n |n\rangle$$

$$\sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n |n\rangle$$

- Hit from left with $\langle m|$:

$$\langle m| \quad \rightarrow \quad \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n |n\rangle$$

$$\sum_{n=0}^{\infty} c_n \sqrt{n}\langle m|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n \langle m|n\rangle$$

$$c_{m+1}\sqrt{m+1} = \alpha\, c_m$$

Continued

$$c_{m+1} = \frac{\alpha}{\sqrt{m+1}} c_m \qquad c_m = \frac{\alpha}{\sqrt{m}} c_{m-1}$$

- Start from: $c_0 = \mathcal{N}(\alpha)$
  - The constant N(α) will be used at the end for normalization
- Try a few iterations:

$$c_1 = \frac{\alpha}{\sqrt{1}} c_0 = \frac{\alpha}{\sqrt{1}} \mathcal{N}(\alpha)$$

$$c_2 = \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha^2}{\sqrt{2\cdot 1}} \mathcal{N}(\alpha)$$

$$c_3 = \frac{\alpha}{\sqrt{3}} c_2 = \frac{\alpha^3}{\sqrt{3\cdot 2\cdot 1}} \mathcal{N}(\alpha)$$

$$c_4 = \frac{\alpha}{\sqrt{4}} c_2 = \frac{\alpha^4}{\sqrt{4\cdot 3\cdot 2\cdot 1}} \mathcal{N}(\alpha)$$

- So clearly by induction we have:

$$c_n = \frac{\alpha^n}{\sqrt{n!}} \mathcal{N}(\alpha)$$

Normalization Constant

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\mathcal{N}(\alpha)$$

- So we have:

$$|\alpha\rangle = \mathcal{N}(\alpha)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

- For normalization we require:

$$\begin{aligned}1 &= \langle\alpha|\alpha\rangle \\ &= |\mathcal{N}(\alpha)|^2 \sum_{\substack{n=0\\m=0}}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|n\rangle \\ &= |\mathcal{N}(\alpha)|^2 \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} \\ &= |\mathcal{N}(\alpha)|^2 e^{|\alpha|^2}\end{aligned}$$

- Which gives us:

$$\mathcal{N}(\alpha) = e^{-\frac{|\alpha|^2}{2}} \qquad |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Orthogonality

- Let us compute the inner-product of two coherent states:

$$\langle \alpha | \beta \rangle = e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{\substack{n=0 \\ m=0}}^{\infty} \frac{\alpha^{*m} \beta^n}{\sqrt{m! n!}} \langle m | n \rangle$$

$$= e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha^* \beta\right)^n}{n!}$$

$$= e^{-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha^* \beta}$$

- Note that:

$$e^{-|\alpha - \beta|^2} = e^{-\left(\alpha^* - \beta^*\right)(\alpha - \beta)}$$

$$= e^{-\left(|\alpha|^2 + |\beta|^2 + a^* \beta + \beta^* \alpha\right)}$$

$$= \left| \langle \alpha | \beta \rangle \right|^2$$

- So coherent states are *NOT* orthogonal
  - Does this contradict our earlier results regarding the orthogonality of eigenstates?

Expectation Values of Position Operator

- Lets look at the shape of the coherent state wavepacket
  - Let $\psi_\alpha(x) = \langle x | \alpha \rangle$

$$\langle X \rangle = \int dx \psi_\alpha^*(x) x \psi_\alpha(x)$$

  - Better to avoid these integrals, instead lets try using A and $A^\dagger$:

$$\langle X \rangle = \langle \alpha | \frac{\lambda}{\sqrt{2}} \left(A + A^\dagger \right) | \alpha \rangle$$

  - Recall the definition of $|\alpha\rangle$:

$$A|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle \alpha | A^\dagger = \alpha^* \langle \alpha |$$

$$\langle X \rangle = \frac{\lambda}{\sqrt{2}} \left(\langle \alpha | A | \alpha \rangle + \langle \alpha | A^\dagger | \alpha \rangle \right)$$

$$= \frac{\lambda}{\sqrt{2}} \left(\alpha \langle \alpha | \alpha \rangle + \alpha^* \langle \alpha | \alpha \rangle \right)$$

$$= \frac{\lambda}{\sqrt{2}} \left(\alpha + \alpha^* \right)$$

$$\langle X \rangle = \sqrt{2} \lambda \mathrm{Re}\{\alpha\}$$

Expectation Value of Momentum Operator

- We can follow the same procedure for the momentum:

$$\langle P\rangle = -i\frac{\hbar}{\sqrt{2}\lambda}\langle\alpha|\left(A - A^{\dagger}\right)|\alpha\rangle$$

$$= \frac{\sqrt{2}\hbar}{2i\lambda}\left(\langle\alpha|A|\alpha\rangle - \langle\alpha|A^{\dagger}|\alpha\rangle\right)$$

$$= \frac{\sqrt{2}\hbar}{2i\lambda}\left(\alpha - \alpha^{*}\right)$$

$$\langle P\rangle = \frac{\sqrt{2}\hbar}{\lambda}\mathrm{Im}\{\alpha\}$$

$$\langle X\rangle = \sqrt{2}\lambda\mathrm{Re}\{\alpha\}$$

- Not surprisingly, this gives:

$$\alpha = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}\langle X\rangle + i\frac{\lambda}{\hbar}\langle P\rangle\right)$$

Variance in Position

- Now let us compute the spread in x:

$$\langle X^2 \rangle = \langle \alpha | \frac{\lambda^2}{2} \left(A + A^\dagger \right)^2 | \alpha \rangle$$

$$= \langle \alpha | \frac{\lambda^2}{2} \left(A^2 + AA^\dagger + A^\dagger A + A^\dagger A^\dagger \right) | \alpha \rangle$$

- Put all of the A 's on the right and the $A^\dagger$'s on the left:
  - This is called 'Normal Ordering'

$$= \langle \alpha | \frac{\lambda^2}{2} \left(A^2 + 2A^\dagger A + 1 + A^\dagger A^\dagger \right) | \alpha \rangle$$

$$= \frac{\lambda^2}{2} \left(\alpha^2 + 2\alpha^* \alpha + 1 + \alpha^{*2} \right)$$

$$= \frac{\lambda^2}{2} \left(\left(\alpha + \alpha^* \right)^2 + 1 \right)$$

$$\langle X \rangle = \frac{\lambda}{\sqrt{2}} \left(\alpha + \alpha^* \right)$$

$$\langle X^2 \rangle = \langle X \rangle^2 + \frac{\lambda^2}{2}$$

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\lambda}{\sqrt{2}}$$

Exactly the same variance as the ground state $|n=0\rangle$

Momentum Variance

- Similarly, we have:

$$\left\langle P^2 \right\rangle = -\frac{\hbar^2}{2\lambda^2}\left\langle \alpha \middle| \left(A - A^\dagger\right)^2 \middle| \alpha \right\rangle$$

$$= -\frac{\hbar^2}{2\lambda^2}\left\langle \alpha \middle| \left(AA - AA^\dagger - A^\dagger A + A^\dagger A^\dagger\right) \middle| \alpha \right\rangle$$

  - Normal ordering gives:

$$\left\langle P^2 \right\rangle = -\frac{\hbar^2}{2\lambda^2}\left\langle \alpha \middle| \left(AA - 2A^\dagger A - 1 + A^\dagger A^\dagger\right) \middle| \alpha \right\rangle$$

$$= -\frac{\hbar^2}{2\lambda^2}\left\langle \alpha \middle| \left(AA - 2A^\dagger A - 1 + A^\dagger A^\dagger\right) \middle| \alpha \right\rangle$$

$$= -\frac{\hbar^2}{2\lambda^2}\left(\alpha^2 - 2\alpha^*\alpha + \alpha^{*2} - 1\right)$$

$$= -\frac{\hbar^2}{2\lambda^2}\left(\left(\alpha - \alpha^*\right)^2 - 1\right)$$

$$\left\langle P \right\rangle = \frac{\sqrt{2}\hbar}{2i\lambda}\left(\alpha - \alpha^*\right)$$

$$\left\langle P^2 \right\rangle = \left\langle P \right\rangle^2 + \frac{\hbar^2}{2\lambda^2}$$

$$\Delta P = \sqrt{\left\langle P^2 \right\rangle - \left\langle P \right\rangle^2} = \frac{\hbar}{\sqrt{2}\lambda}$$

Minimum Uncertainty States

- Let us check what Heisenberg Uncertainty Relation says about coherent states:

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \frac{\lambda}{\sqrt{2}}$$

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \frac{\hbar}{\sqrt{2}\lambda}$$

$$\Delta X \Delta P = \frac{\lambda}{\sqrt{2}} \frac{\hbar}{\sqrt{2}\lambda}$$

$$\Delta X \Delta P = \frac{\hbar}{2}$$

- So we see that all coherent states (meaning no matter what complex value α takes on) are *Minimum Uncertainty States*
  - This is one of the reasons we say they are ‘most classical’

Time Evolution

- We can easily determine the time evolution of the coherent states, since we have already expanded onto the Energy Eigenstates:
  - Let

$$\left|\psi(t=0)\right\rangle = \left|\alpha_0\right\rangle$$

  - Thus we have:

$$\left|\psi(0)\right\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \left|n\right\rangle$$

$$\left|\psi(t)\right\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega(n+1/2)t} \left|n\right\rangle$$

$$= e^{-i\omega t/2} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\omega n t} \left|n\right\rangle$$

$$= e^{-i\omega t/2} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha_0 e^{-i\omega t}\right)^n}{\sqrt{n!}} \left|n\right\rangle$$

  - Let

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

$$\psi(t) = \left|\alpha(t)\right\rangle$$

By this we mean it remains in a coherent state, but the value of the parameter α changes in time

Why 'most classical'?

- What we have learned:
  - Coherent states remain coherent states as time evolves, but the parameter α changes in time as

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

  - This means they remain a minimum uncertainty state at all time
  - The momentum and position variances are the same as the $n=0$ Energy eigenstate
  - Recall that:

$$\langle X \rangle = \sqrt{2}\lambda \mathrm{Re}\{\alpha\}$$

$$\langle P \rangle = \frac{\sqrt{2}\hbar}{\lambda} \mathrm{Im}\{\alpha\}$$

  - So we can see that:

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(\frac{x_0}{\lambda} + i\frac{\lambda}{\hbar}p_0\right)$$

$$x_0 = \langle \alpha(t) | X | \alpha(t) \rangle$$

$$p_0 = \langle \alpha(t) | P | \alpha(t) \rangle$$

  - We already know that <X> and <P> behave as classical particle in the Harmonic Oscillator, for any initial state.

$$x(t) = x_0 \cos(\omega t) + \frac{p_0}{\omega}\sin(\omega t) \qquad p(t) = p_0 \cos(\omega t) - \omega x_0 \sin(\omega t)$$

Conclusions

- The Coherent State wavefunction looks exactly like ground state, but shifted in momentum and position. It then moves as a classical particle, while keeping its shape fixed.
  - Note: the coherent state is also called a ‘Displaced Ground State’