Chapter 5

Hedging in continuous time model

In the binomial tree model, we are led to model the returns from a stock as

$$\frac{S_{t+h} - S_t}{S_t} \simeq \mu h + \sigma\sqrt{h} \tag{5.0.1}$$

We may like to find the continuous version of (5.0.1).

From the binomial model tree with drift given by equation (5.0.1), we could guess that

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \tag{5.0.2}$$

is a reasonably similar model. In fact, this model is the continuous time analogue of the binomial tree.

We have also proved that under the RNPM the binomial model converges to the Black–Scholes model and its dynamic is given by

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t \tag{5.0.3}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under the RNPM.

5.0.1 Delta Hedging with Black–Scholes Model

Problem 1. A bank has sold for \$300000 a European call option on 100000 shares of a non–dividend paying stock $S_0 = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$, $T = 20$ weeks. The Black–Scholes value of the option is \$240,000

Question: How does the bank hedge its risk?

Naked and Covered Positions :

1. Take no action: exposed to lose if stock increases

2. Buy 100000 shares today exposed to lose if stock decreases

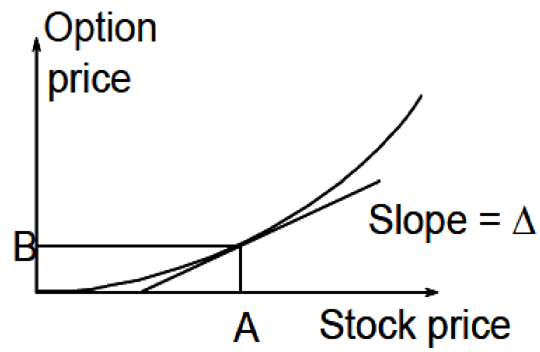
Both positions expose the bank to risk

Solutions:

1. **Stop–Loss Strategy**: Buying 100000 shares as soon as price reaches \$50. Selling 100000 shares as soon as price falls below \$50.
   This deceptively simple hedging strategy does not work well.

2. **Delta:** Delta (Δ) is the rate of change of the option price with respect to the underlying

$$\Delta = \frac{\partial C}{\partial S} \text{ where } C \text{ is the price of the option}$$



Delta of the call

$$\Delta_{call} = \frac{\partial C}{\partial S} = \mathbf{N}(d_+)$$

Properties : $0 < \Delta_{call} < 1$ the mapping $S \longmapsto \Delta_{call}$ is increasing.

3. **Delta Neutral Hedge:** Set delta of portfolio equal to zero. This requires that you take offsetting positions, protected against a small change in the stock price

4. **Delta Neutral Hedge for Call Option:**

   (a) Buy β shares of stock

   (b) Sell one call

   Portfolio Value: $V = \beta S - C$
   Set up hedge: $\frac{\partial V}{\partial S} = \beta - \frac{\partial C}{\partial S} = \beta - \frac{\partial C}{\partial S} = 0$ evaluated at current stock price. Therefore, $\beta = \Delta$.
   Hedge must be adjusted with as stock price changes
   What should you do if S increases?
   What should you do if S falls?
   Buy high, sell low strategy

Example 5.0.1 *Assume that* $C = 10$, $S = 100$, *and* $\Delta = 0.75$. *If stock price goes up by* \$1, *by approximately how much will the call increase?*
Answer: \$0.75.
If you write one call, how many shares of stock must you own so that a small change in the stock price is offset by the change in the short call?
Answer: 0.75 *shares.*

5.0.2 Delta Neutral Hedge with Puts

Black–Scholes Put Delta: Delta of the put

$$\Delta_{put} = \frac{\partial P}{\partial S} = \mathbf{N}(d_+) - 1 = \Delta_{call} - 1$$

Properties : $-1 \leq \Delta_{put} \leq 0$ the mapping $S \longmapsto \Delta_{put}$ is decreasing.
Delta Neutral Hedge
* Buy $|\Delta|$ shares of stock
* Buy one put

Other "Greeks" Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset:

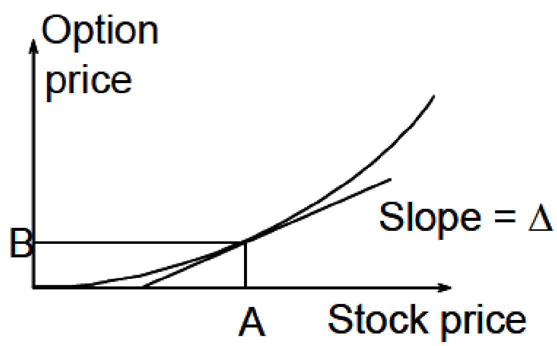
$$\Gamma_{call} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_{call}}{\partial S} = \frac{\partial \mathbf{N}\left(d_+(t)\right)}{\partial S} = \mathbf{N}'\left(d_+(t)\right)\frac{\partial d_+(t)}{\partial S} = \frac{\mathbf{N}'\left(d_+(t)\right)}{S_t \sigma \sqrt{T-t}} > 0$$

This implies the convexity of the call option price C_t with respect to the underlying (S_t).

$$\Gamma_{put} = \frac{\partial^2 P}{\partial S^2} = \frac{\partial \Delta_{put}}{\partial S} = \left(\frac{\partial\left(\mathbf{N}\left(d_+(t)\right) - 1\right)}{\partial S}\right) = \frac{\partial \mathbf{N}\left(d_+(t)\right)}{\partial S} = \Gamma_{call} = \frac{\mathbf{N}'\left(d_+(t)\right)}{S_t \sigma \sqrt{T-t}} > 0$$

This implies also that the put option price P_t is convex with respect to the underlying (S_t).

* Gamma Addresses Delta Hedging Errors Caused By Curvature



5.1 Delta hedging the Black–Scholes framework

5.1.1 Dividend paying stock

The Goals of this section is to introduce Greek parameters describing the sensitivities of the a price option with respect to the parameters of the model.

Introduce Delta Δ and dynamic delta hedging:

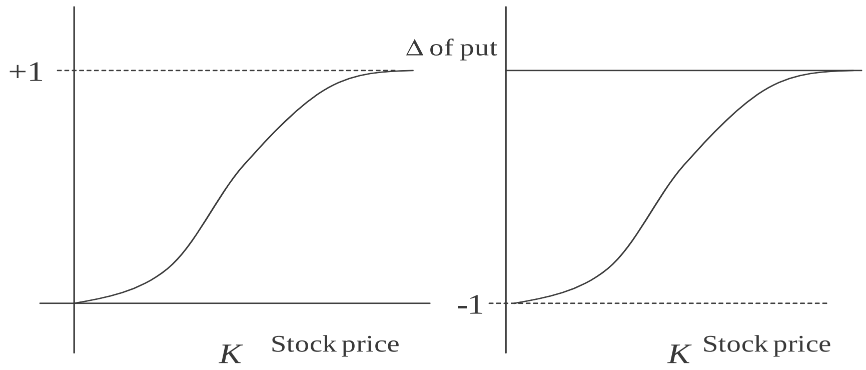
Delta is the slope (first derivative) of the P&L/underlying curve. A delta hedge protects only against small movements in the price of the underlying. An example of a delta hedge is when you buy a put, which gives you negative delta and positive gamma, and then buy enough of the underlying to zero out your total delta. This hedge does not protect against larger movements of the underlying. When the underlying moves, the non-zero gamma will change your delta, causing you to need to re-hedge. Many people mistakenly call this re-hedging "gamma hedging", but it is not the case; it is just dynamic hedging of delta in reaction to gamma.

5.2 Delta and Dynamic (Delta hedging)

1. **Naked position: no hedge strategy**:

   (a) Take no action and maintain the naked position: that is wait for expiry and **hope for the best**. This is exposed to lose if stock increases.

      i. It the call is In The Money (ITM) ($S_T > K$) at time T the bank needs to sell 100000 shares to the call holder for the predetermined price $K = 50$ dollars per share, hence the banks loses $S_T - K$ dollars per share. This loss could be high.



ii. It the call is Out of The Money (OTM) ($S_T < K$) at time T the call holder will not exercise his right, thus the banks needs to do nothing. The bank can earn the call premium 300000 dollars, which is received at time zero.

2. **Fully covered hedge strategy**

(a) Buy 100000 shares today at $S_0 = 49$ dollars per share

i. It the call is In The Money (ITM) ($S_T \geq K$) at time T the bank sells 100000 shares to the call holder for the predetermined price $K = 50$ dollars per share, hence the bank can earn $K - S_0 = 50 - 49 = 1$ dollars per share minus the interest cost to purchase 100000 shares at $S_0 = 49$. Notice also that if $S_0 > K$ the bank will suffer a loss definitely.

ii. It the call is Out of The Money (OTM) ($S_T < K$) at time T the call holder will not exercise his right, thus the banks needs to do nothing. The bank can earn the call premium 300000 dollars, but the stock shares position could suffer a large loss if $S_T < S_0$.

Both the above two strategies leave the bank exposed to significant risk.

Neither a naked position nor a covered position provides a satisfactory hedge.

call–put parity shows that the exposure from writing a covered call is the same as the exposure from writing a naked put.

For a perfect hedge the standard deviation of the cost of writing and hedging the option is zero.

Stop–Loss Strategy

This involves:

Buying 100,000 shares as soon as price reaches \$50

Selling 100,000 shares as soon as price falls below \$50

This deceptively simple hedging strategy does not work well in practice:

Purchases and subsequent sales cannot be made at K. Transactions costs could easily consume the option premium and then some.

Delta hedging

Delta is very closely related to the idea of the replicating portfolio intuition. Most traders use more sophisticated hedging schemes. These involve calculating measures such as delta.

This involves maintaining a delta neutral portfolio

The delta of a European call on a stock paying dividends at rate q is $e^{-q(T-t)}\mathbf{N}(d_+(t))$.

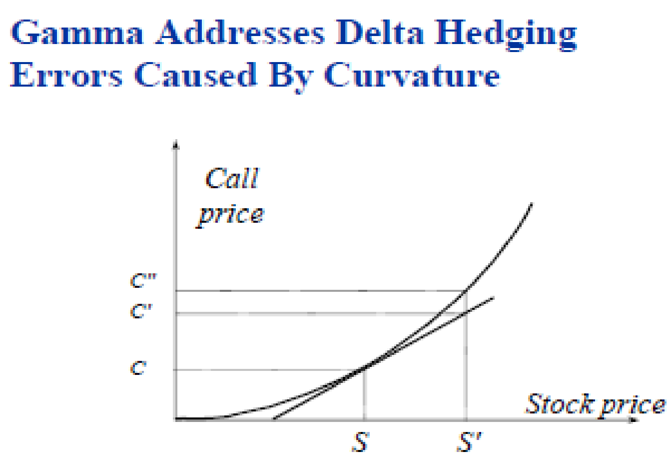
The delta of a European put is $e^{-q(T-t)}\left(\mathbf{N}(d_+(t)) - 1\right)$

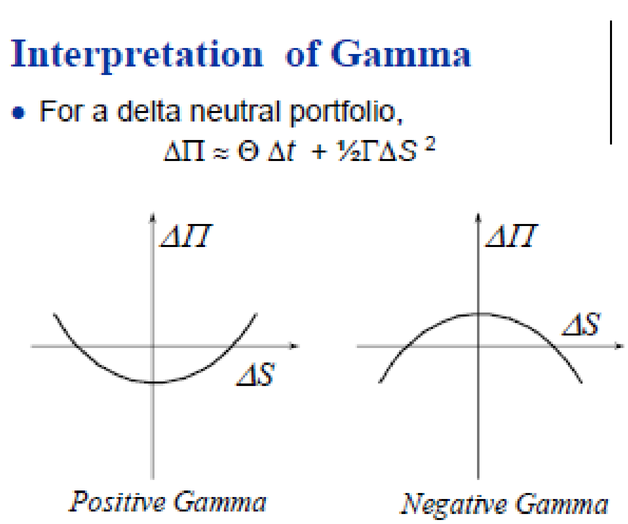
The hedge position must be frequently rebalanced. Delta hedging a written option involves a "buy high, sell low" trading rule (see spreadsheet in the excel file).

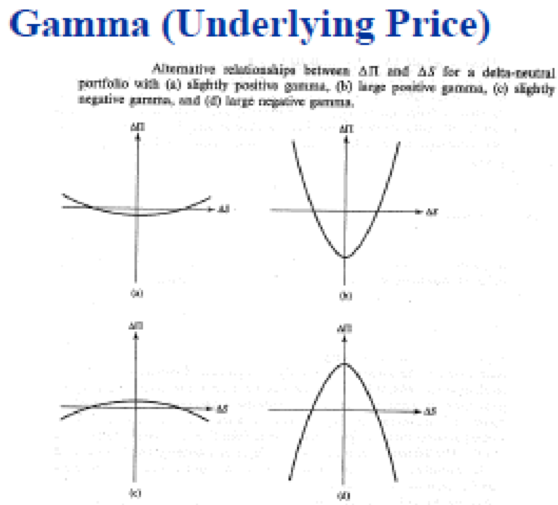
Gamma Γ and Theta Θ

Gamma is the second derivative of the P&L/underlying curve. A gamma hedge protects only against small movements of gamma; gamma will move when either the underlying or its implied volatility move. An example of a gamma hedge is when you buy a put, which gives you negative delta and positive gamma, then sell a call to zero out your gamma but give you even more negative delta. This exposes you to large movements of the underlying, so you will likely want to then buy enough of the underlying to zero out your delta. A gamma hedge does not protect against larger movements of gamma, because the put and call each have non-zero "speed"

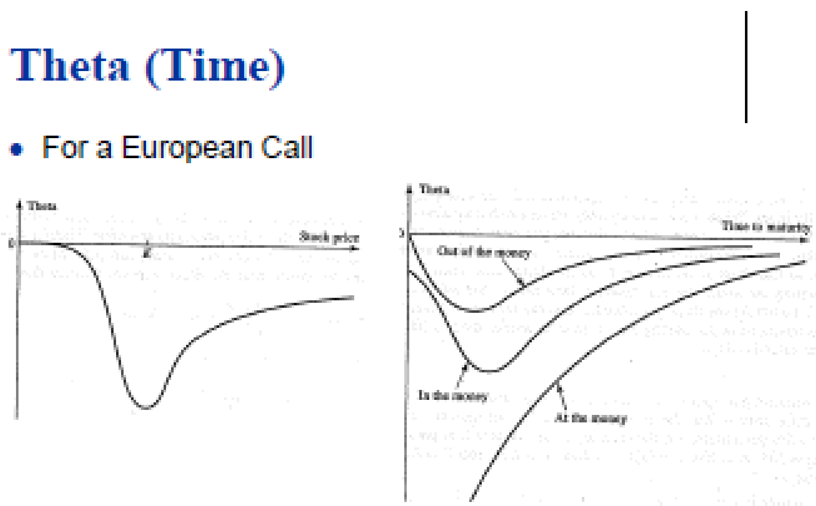
Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset see the next figure for the variation of Γ with respect to the stock price for a call or put option







Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time; The following figure shows the variation of Θ with respect to the stock price for a European call.



Theta (Time): In general Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value of the derivative with respect to the time to maturity.

$$Th\hat{e}ta_{Call} = \Theta_{call} = \frac{\partial C_t}{\partial (T-t)} = -\frac{\sigma S_t}{2\sqrt{T-t}}\mathbf{N}'(d_+(t)) - Kre^{-r(T-t)}\mathbf{N}(d_-(t)).$$

The theta for the put can be deduced from the call–put parity formula:

$$C_t + Ke^{-r(T-t)} = P_t + S_t,$$

hence

$$\begin{aligned} Th\hat{e}ta_{Put} &= \Theta_{put} = \frac{\partial P_t}{\partial (T-t)} = \frac{\partial C_t}{\partial (T-t)} - rKe^{-r(T-t)} \\ &= -\frac{\sigma S_t}{2\sqrt{T-t}}\mathbf{N}'(d_+(t)) + Kre^{-r(T-t)}\mathbf{N}(-d_-(t)). \end{aligned}$$

Properties: The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of the option declines.

We can develop a closed form for theta of a European Call as we did for delta; from the Black–Scholes equation.

Consider a call option on a non–dividend–paying stock where the stock price is \$49, the strike price is \$50, the risk–free rate is 5%, the time to maturity is 20 weeks (= 0.3846 years), and the volatility is 20%.

Question: Find the gamma and the theta call option.

Answer: In this case, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.3846$. The theta of the call option is then -4.31. Then is the theta is $\frac{-4.31}{365} = -0.0118$ per calendar day, or $\frac{-4.31}{252} = -0.0171$ per trading day.

The option's gamma is 0.066 this means that when the stock price changes by δS, the delta of the option changes by $0.066\delta S$.

5.2.1 Relation ship between Δ, Γ and Θ

The framework of the Black–Scholes model the greek parameters Delta, Theta and Gamma are linked by the following famous equation

$$\Theta_t + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t + r S_t \Delta_t - r C_t = 0.$$

called the Black–Scholes partial differential equation B–S (PDE).

In general the price of a single derivative on a q–dividend paying stock must satisfy the Black–Scholes PDE. If Π_t stands for the price of a single derivative of European type at time t must also satisfy the following PDE

$$\frac{\partial \Pi_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \Pi_t}{\partial^2 S} + (r - q)\, S_t \frac{\partial \Pi_t}{\partial S} - r\Pi_t = 0.$$

If we use the Greeks notations we get

$$\Theta_t + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t + (r - q)\, S_t \Delta_t - r\Pi_t = 0$$

For a delta hedged portfolio $\Delta = 0$, then we have

$$\Theta_t + \frac{1}{2}\sigma^2 S_t^2 \Gamma_t = r\Pi_t$$

So when theta is large and positive, gamma should be large and negative and vice versa.

Suppose that δS is the price change of an underlying asset during a small interval of time, δt, and $\delta \Pi_t$ is the corresponding price change in the portfolio, if terms of order higher than δt are ignored, then

$$\delta \Pi_t = \Theta_t \delta t + \frac{\Gamma_t}{2} \delta S_t^2 \tag{5.2.4}$$

Suppose that the gamma of a delta-neutral portfolio of options on an asset is -10000. Equation (5.2.4) shows that, if a change of $+2$ or -2 in the price of the asset occurs over a short period of time, there is an unexpected decrease in the value of the portfolio of approximately $0.5 \times 10000 \times 2^2 = \$20000$.

Vega $\mathcal{V}$ and Rho ρ

Vega $\mathcal{V}$ is the measurement of an option's sensitivity to changes in the volatility of the underlying asset. Vega represents the amount that an option contract's price changes in reaction to a 1% change in the implied volatility of the underlying asset. Volatility measures the amount and speed at which price moves up and down, and is often based on changes in recent, historical prices in a trading instrument.

$$\begin{aligned} Vega_{Call} &= \frac{\partial C_t}{\partial \sigma} = S_t \sigma \sqrt{T - t}\mathbf{N}'(d_+(t)). \\ Vega_{Put} &= \frac{\partial P_t}{\partial \sigma} = -S_t \sigma \sqrt{T - t}\mathbf{N}'(-d_+(t)). \end{aligned}$$

Rho (ρ) is the rate of change of the value of a derivative with respect to the interest rate. For the European options ρ has a closed form which is given by:

$$\begin{aligned} Rho_{Call} &= \rho_{call} = \frac{\partial C_t}{\partial r} = (T-t)Ke^{-r(T-t)}\mathbf{N}(d_+(t)). \\ Rho_{Put} &= \rho_{put} = \frac{\partial P_t}{\partial r} = -(T-t)Ke^{-r(T-t)}\mathbf{N}(-d_-(t)). \end{aligned}$$

For currency options there are 2 rhos. The second one is given by Φ.

$$\begin{aligned} Phi_{Call} &= \Phi_{call} = \frac{\partial C_t}{\partial r_f} \\ Phi_{Put} &= \Phi_{put} = \frac{\partial P_t}{\partial r_f}. \end{aligned}$$

Summary

Greek letters under the classical (constant parameters) Black–Scholes model for European options on an asset that provides a yield at rate q.

Greek letter	Call option	Put option
Delta	$e^{-q(T-t)}\mathbf{N}(d_+(t))$	$e^{-q(T-t)}\left(\mathbf{N}(d_+(t))-1\right)$
Gamma	$\dfrac{e^{-q(T-t)}\mathbf{N}'\left(d_+(t)\right)}{S_t\sigma\sqrt{T-t}}$	$\dfrac{e^{-q(T-t)}\mathbf{N}'\left(d_+(t)\right)}{S_t\sigma\sqrt{T-t}}$
Theta	$-\dfrac{\sigma S_t\mathbf{N}'(d_+(t))}{2\sqrt{T-t}} + qS_t\mathbf{N}(d_+(t))e^{-q(T-t)}$ $-Kre^{-r(T-t)}\mathbf{N}(d_-(t))$	$-\dfrac{\sigma S_t\mathbf{N}'(d_+(t))}{2\sqrt{T-t}} - qS_t\mathbf{N}(-d_+(t))e^{-q(T-t)}$ $+Kre^{-r(T-t)}\mathbf{N}(-d_-(t))$
Vega	$S_t\sigma\sqrt{T-t}\mathbf{N}'(d_+(t))e^{-q(T-t)}$	$-S_t\sigma\sqrt{T-t}\mathbf{N}'(-d_+(t))e^{-q(T-t)}$
Rho	$(T-t)Ke^{-r(T-t)}\mathbf{N}(d_+(t))$	$-(T-t)Ke^{-r(T-t)}\mathbf{N}(-d_-(t))$
Phi		

where

$$d_{\pm}(t) = \frac{\ln(\frac{S_t}{K}) + (r-q+\frac{\sigma^2}{2})\,(T-t)}{\sigma\sqrt{T-t}}$$

5.2.2 Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is S is

$$\frac{\partial \Pi_t}{\partial S}$$

where Π_t is the value of the portfolio at time.

The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity α_i of option i ($1 \le i \le n$), the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} \alpha_i \Delta_i$$

where Δ_i, is the delta of the i^{th} option. The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being delta neutral.

Suppose a financial institution has the following three positions in options on a stock:

1. A long position in 100000 call options with strike price \$55 and an expiration date in 3 months. The delta of each option is 0.533.
2. A short position in 200000 call options with strike price \$56 and an expiration date in 5 months. The delta of each option is 0.468.
3. A short position in 50000 put options with strike price \$56 and an expiration date in 2 months. The delta of each option is -0.508.

The delta of the whole portfolio is

$$100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) = -14900$$

This means that the portfolio can be made delta neutral by buying 14900 shares.